Computational Modeling of 2D Motion

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Students investigate a computational model of projectile motion by comparing its results to results obtained through experiment. On-level students do this for a 'standard' angled projectile, verifying that the angle producing maximum range is 45˚, deriving the range formula, and using kinematics equations to develop displacement functions for the projectile's motion. Honors/AP students complete the same exercise while accounting for the initial height of the projectile from the surface. The activity culminates in a lab report.

Problem
How do experimental data inform theoretical models?

Objectives
- To investigate the relationship between the initial angle and range of a projectile
- To compare computer-generated data to measured data
- To define and construct a model of projectile motion

Anticipated Learner Outcomes
After completing this lesson, students will be able to:
- relate problem-solving to model construction
- explain the difference between experiment and theory in a concrete way
- express equations as graphical functions
- derive the 'range formula'
- state the relationship between angle and range both experimentally and mathematically

Standards
<u>Common Core Standards</u>
Standards for Literacy in Science and Technical Subjects 6-12
- Key Ideas and Details (11-12) 3: Follow precisely a complex multistep procedure when carrying out experiments, taking measurements, or performing technical tasks; analyze the specific results based on explanations in the text
- Craft and Structure (11-12) 4: Determine the meaning of symbols, key terms, and other domain-specific words and phrases as they are used in a specific scientific or technical context relevant to *grades 11-12 texts and topics*.
- Integration of Knowledge and Ideas (11-12) 9: Synthesize information from a range of sources (e.g., texts, experiments, simulations) into a coherent understanding of a process, phenomenon, or concept, resolving conflicting information when possible.

<u>Georgia Performance Standards</u>
SCSh3. Students will identify and investigate problems scientifically.

SCSh4. Students will use tools and instruments for observing, measuring, and manipulating scientific equipment and materials.

SCSh5. Students will demonstrate the computation and estimation skills necessary for analyzing data and developing reasonable scientific explanations.

SCSh6. Students will communicate scientific investigations and information clearly.

SCSh8. Students will understand important features of the process of scientific inquiry.

SP1. Students will analyze the relationships between force, mass, gravity, and the motion of objects.

a. Calculate average velocity, instantaneous velocity, and acceleration in a given frame of reference.
b. Compare and contrast scalar and vector quantities.
c. Compare graphically and algebraically the relationships among position, velocity, acceleration, and time.
f. Measure and calculate two-dimensional motion (projectile and circular) by using component vectors.

Background

In high school physics courses, much emphasis is placed on problem solving. Students are essentially trained to treat physical models as lists of equations that they pick and choose from to solve basic single-answer problems. There's nothing particularly incorrect with this approach, but it is lacking for students who choose to pursue engineering or the physical sciences for careers. This is because practicing physicists are in the business of constructing and analyzing *models*, meaning that the end result is very different.

There are several proponents in the education community suggesting that high school physics courses be entirely restructured on a model-based approach. Implementing this approach can be difficult in standards-based courses with large amounts of content due to the sheer amount of time it takes.

In a model-based approach, students complete units in two stages. First, they design and implement experiments to help them build a model – to essentially develop the equations that would be delivered by the instructor in a problem-based course. Second, after the model has been properly refined, it is used to develop and solve problems. Particular emphasis is placed on realistic open-ended problems when using models, and simply giving students lists of problems to solve is actively discouraged.

It is easy to see, then, that where studying the three constant acceleration kinematics equations would take 3-6 days of instruction in a traditional problem-based classroom, it could take several weeks in a model-based classroom. This would certainly be time well-spent, but in a course with a significant amount of content that is required to cover, it is simply not practical.

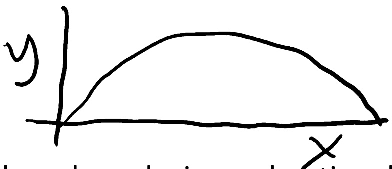
This lesson, then, is a hybrid of model- and problem-based learning. It uses technology to introduce students to the idea that physics equations are more properly thought of as functions rather than 'answer generators.' Students are not expected to construct the entire model from scratch, but instead are asked to use their knowledge to determine the equations used in the model. Finally, experiment is used to verify the model, rather than to derive it.

The purpose of this is to help students connect the equations used in class to the lab exercises conducted. Students often struggle to see how lab activities fit into a problem-based classroom. There are a variety of reasons for this, but one way to confront this conflict is to compare theoretical data generated from a set of equations to experimental data generated in a lab exercise. The goal is for students to realize that the model *does* match the lab data in significant ways and to begin understanding that physicists value equations over 'answers' to specific problems.

The specific system chosen to model is projectile motion, which is a well-understood problem. Conceptual/on-level students will consider a projectile that starts and lands at the same height, giving us the position equations:

$$x(t) = (v_0 \cos\theta)t$$

$$y(t) = (v_0 \sin\theta)t - \frac{1}{2}gt^2$$

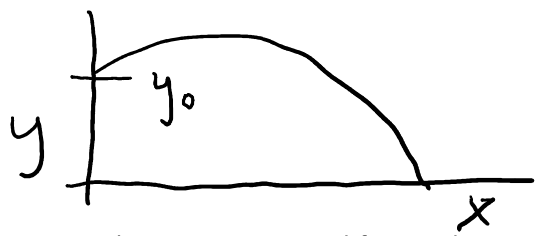


where v_0 is the initial launch velocity, θ is the initial angle of launch, and g is acceleration due to gravity. What the model does is solves these equations for a finite number of time steps and then plots the x and y values for the student, showing them the trajectory of the projectile. The model allows the user to vary the constant values v_0, θ, and g, as well as the time step Δt, which is related to the number of points calculated – a big Δt will only calculate a few points, giving a crude graph; a larger Δt will calculate more points, giving a more detailed graph.

The only difference for honors/AP students is that they will be more realistic about the projectile problem's initial set-up – the projectile's initial height is higher than its final height. This only alters the *y*-direction function:

$$x(t) = (v_0 \cos\theta)t$$

$$y(t) = y_0 + (v_0 \sin\theta)t - \frac{1}{2}gt^2$$



where y_0 is the initial height of the projectile.

The second part of the model involves the maximum range value. It is expected for students to understand that due to the component nature of the initial velocity, an ideal projectile's maximum range occurs when the initial launch angle is 45˚. It is a useful exercise in all classes to derive the 'range formula,' although it is recommended that the instructor verifies that students are familiar with the trig identity $\sin(2\theta) = 2\sin\theta\cos\theta$ before beginning.

For classes containing students with math challenges, the author recommends completing the derivation using whole class instruction. Otherwise, the derivation should be completed by each individual lab group. On-level students should derive:

$$R = \frac{{v_0}^2 \sin(2\theta)}{g}$$

For honors/AP students, the derivation is somewhat more complex. It is *strongly* recommended that honors/AP students complete the 'equal heights' range derivation before attempting the 'unequal heights' formula. The formula is best presented as two:

$$R = (v_0 \cos\theta)t_{range}$$

where

$$t_{range} = \frac{1}{g}\left(v_0 \sin\theta + \sqrt{2gy_0 + (v_0 \sin\theta)^2}\right)$$

The final piece that needs to be addressed prior to beginning the exercise is a basic overview of Excel macros. It should be noted that this simulation can be written using any programming language or scripting software (Matlab, Mathematica, Maple, etc.), but the author chose Excel because of its wide availability.

Excel uses modified Visual Basic for its macro language, so any teachers or students familiar with Visual Basic should be able to read the macro easily. However, knowledge of programming is **not required** to complete this lesson – the macro has already been written.

To access the macro, open the provided file (either On-Level Projectile Model.xlsm or Honors-AP Projectile Model.xlsm) and go to the 'View' tab at the top of the screen. On the right side of this tab, there is a button for 'Macro.' Clicking on this button will bring up a dialog box that allows you to either run or edit the macro 'Projectile.'

Select 'Edit' and the VBA editor will open, allowing you to view the code directly. Note that it is all commented for readability. You are only interested in three lines – look at the top of the editor and it will tell you the line number you're on. Lines 30 and 31 contain the x and y functions, and line 65 contains the range function on the on-level file (lines 68 and 69 contain the range function on the honors-AP file).

Before beginning the lesson, you need to DELETE these functions (leave the "x =", "y =", "range =" parts so students know where to type things in!!) – this is where students will go in to enter their functions.

Note the odd syntax for the honors-AP range function – it turns out that Excel doesn't have a 'square root' in its macro library. You may want to remind your students that instead of using a square root, they can use the '1/2' power.

To run the file, go back to the spreadsheet and select 'Run' instead of edit. The columns should automatically populate and the graphs should pop up. The code will ask you if you want to run range data or not. This is relevant later in the lesson.

Materials and Supplies
Students will need access to a computer with Excel and a word-processing program. For the lab, students will need meter sticks or other range measuring device, a protractor, and a launching mechanism of some sort:

- Arbor Scientific has an air powered device that goes long ranges: http://www.arborsci.com/air-powered-projectile
- PASCO offers a short-range device for classroom use: http://www.pasco.com/prodCatalog/ME/ME-6800_projectile-launcher-short-range/
- The author of this lesson uses toy rocket launchers that are of a design similar to the one shown at http://www.mightytoy.com/Prop-Shots-Giant-Whistle-Rocket/dp/B003ATBRQO

Plan
Prior to completing this lesson, it is assumed that students have already been introduced to two-dimensional projectile motion and are comfortable with the constant acceleration kinematics formulas. Before beginning the lab activity, it would be appropriate to solve some basic projectile problems.

<u>On-level Course</u>
Stage I (~40-50 minutes)
Working in groups of 2-3, students are expected to complete part one of the handout. They need access to computers so that they can run the Excel file. Basically, students solve projectile problems

traditionally, use the Excel file to investigate those same problems, and then draw conclusions about models.

**At this point, the class should gather together so that the teacher can spend the last 5-10 minutes defining and discussing mathematical modeling. Students should discuss the benefits and limitations of the Excel-based projectile model

Stage II (~40-70 minutes)

**Note: Time is extended because of the range formula derivation – this tends to be an extremely time-consuming part of the exercise.

Working in the same groups, students are expected to complete part two of the handout. This is where they perform the lab. They're asked to design their own procedures, perform them, and plot their data. It is important that they get a time measurement to calculate the projectile's initial velocity. They still need access to the Excel file so that they can obtain the theoretical data points to put on their graphs.

**Graphs should be spot-checked by the teacher to head off any huge mistakes, but they should proceed to the third part without a whole group discussion.

Stage III (~10-15 minutes of class time – conclusion should be assigned as homework at teacher's discretion)

Writing the conclusion can be done outside of class if needed. If done outside of class, some sort of peer grading is recommended in lieu of a whole group summary discussion at the end of the lesson. If done in class, the last 5-10 minutes of time should be devoted to a whole class discussion about range and modeling projectiles. Put particular emphasis on the role of air resistance.

<u>Honors/AP Course</u>

Stage I (~40-50 minutes)

Working in groups of 2-3, students are expected to complete the Modeling Projectile Motion. They need access to computers so that they can run the Excel file. Basically, students solve projectile problems traditionally, use the Excel file to investigate those same problems, and then draw conclusions about models.

**At this point, the class should gather together so that the teacher can spend the last 5-10 minutes defining and discussing mathematical modeling. Students should discuss the benefits and limitations of the Excel-based projectile model

Stage II (~40-70 minutes)

**Note: Time is extended because of the range formula derivation – this tends to be an extremely time-consuming part of the exercise.

Working in the same groups, students are expected to complete part two of the handout. This is where they perform the lab. They're asked to design their own procedures, perform them, and plot their data. It is important that they get a time measurement to calculate the projectile's initial velocity. They still need access to the Excel file so that they can obtain the theoretical data points to put on their graphs.

The lab report should be assigned as homework. It is recommended that no more than one week is given to complete the report; less if class time is used. At the teacher's discretion, groups can write a common report.

Assessment/Rubrics

The major lesson assessment is grading the handout/report itself.

<u>Solutions to On-Level handout</u>

1. 0.990 m
2. 7.59 cm
3. 52.6 cm
4. $x(t) = 9.396\ t$
5. $y(t) = 3.21\ t - 4.9\ t^2$
6. The solutions are stated in the background section
7. Various answers – the gist should be that both methods use the same equations
8. Probably 'worse', because it's not as detailed. 'Better' could be because it runs faster, though.
9. Looking at x-axis instead of y-axis
10. Higher in y, farther in x, longer time
11. Various answers – something should be mentioned about how models give us a lot more data than single calculations
12. (a) Various

    (b) A major limitation should be difficulty/equipment. Anything they come up with is going to be a major pain to actually do.

    (c) It should be

14. Procedure should involve multiple trials, a minimum of 7 angles, and getting the initial velocity from timing the rocket going straight up in the air
23. Answers will vary depending on how meticulously students collected data.
24. A few are: consistency of initial launch, air resistance effects, maintaining initial angle, any sideways motion or bouncing of the projectile, accuracy of range measurement itself depending on how long it is
25. Students should realize that creating an accurate model from this experimental data would be extremely difficult. Other responses will vary appropriately.

<u>Solutions to Honors/AP Handout</u>

1. 0.990 m
2. 7.59 cm
3. 0.816 m
4. The solutions are stated in the background section
5. The calculated answers should be related to specific data points on the graph. It should also be noted that the initial height in #2 must be set to zero.

6. Things like runtime and detail of graph should be mentioned. Actual responses will vary.
7. It should be observed that the laws of physics are the same throughout the universe – treating *g* as a variable gives us a way to study projectiles on other planets, etc.
8. Various answers – something should be mentioned about models providing more data and being more generalized
9. (a) Various

   (b) A major limitation should be difficulty/equipment. Anything they come up with is going to be a major pain to actually do.

   (c) It should be

Suggested Rubric for AP/Honors Lab Report

10% Report is properly formatted, including references
10% Abstract
20% Introduction
20% Methodology
20% Data and Results
20% Conclusion

Summary

The goal of this lesson is threefold: first, to teach the specific content of the angle-dependence of a projectile's range; second, to expose students to the idea of computational modeling in physics; third (and most important), to connect the theoretical models we build using equations to the data collected in experiments. Projectile models are extremely useful in doing this due to the large number of discrepant events that can occur – difficulties in the equipment itself combined with non-negligible air resistance effects make for a rich discussion of both the benefits *and* limitations of theoretical models.

Supplemental Materials

Appendix A: Handout for on-level course
Appendix B: Handouts for honors/AP course

Appendix A

Names: ____

LAB: Exploring Projectile Motion

Part One: Prelab Activity

To review projectile motion, solve the following problems with your partner(s). We will go over the answers before starting the lab.

1. A ball rolls off a 0.78-m high lab table and strikes the floor 0.55-m from the edge of the table. What was the initial velocity of the ball?

2. A toy rocket launcher has an initial launch speed of 2.44 m/s. If the launcher is angled at 30˚, how high in the air will the toy rocket go?

3. The launcher from the previous problem is now angled at 60˚. What is the horizontal range of the toy rocket at this angle?

Instead of using our formulas just to solve problems, let's think of them as *functions* instead.

4. A projectile has an initial velocity of 10 m/s and an initial launch angle of 20˚. Using our equations of motion, write an expression of the *horizontal displacement* of our projectile as a function of *time*.

5. A projectile has an initial velocity of 5 m/s and an initial launch angle of 40˚. Using our equations of motion, write an expression of the *vertical displacement* of our projectile as a function of *time*.

6. Let's make this general. A projectile has an initial velocity of v_0 and an initial launch angle of θ. Write expressions for both its horizontal AND vertical displacements as a function of time.

Take your paper to the teacher to check. DO NOT MOVE ON UNTIL YOUR ANSWERS TO #6 HAVE BEEN INITIALED!

Now that you have your *x*- and - and *y*-functions, it's time to log on to your computer. Open the Excel file that the teacher has given you access to.

This Excel file uses a special program called a 'macro.' What a macro basically does is allows a user to write out all of the math in advance and run it over and over without having to retype anything.

Our macro does two things, but we're only going to use one of them for now. If there's a yellow bar across your screen telling you that the macro has been disabled, the first thing you need to do is click 'Enable' so that the program works properly.

Notice the top left-hand corner of the screen – it lists four variables:

- v_0 is the initial velocity. Set it equal to 2.44, like in #2
- *g* is the acceleration due to gravity. We're on Earth
- θ is the initial angle. Set it equal to 30˚, like in #2
- *Δt* is called the **time step** (basically, how often do you want the macro to calculate data points). Set it equal to 0.01, so that every second, it will calculate 100 times

Go to the “View” tab. On the far right of the menu bar, you should see a button that says “Macros.” Click on it. A box pops up listing all the macros in this file (there’s only one). Click “Edit.”

The macro in this file is incomplete! It’s up to you to enter the parameters so that it will calculate everything properly.

Go to **Line 30** – the formulas for *x* and *y* are missing! Using your equations from #6, fix the macro so that it will run.

When you’re done, go to “File” and select “Save and Exit to Excel.” You should be back to the spreadsheet. Click on “Macros” again, but now select “**Run**” in the box.

When it asks you if you want to calculate the range for all angles, select “No.” We’ll go back to that later.

7. Look at your Y vs. X graph. Check and make sure that your answer for #2 matches the data you’ve just created. In the space below, explain how you determine whether your data matches and why we expect it to.

8. Change the time step to 0.1 and run the macro again. Is this data better or worse than your previous data? Explain your reasoning (and what you mean by ‘better or worse’).

9. Check your answer for #3 using our macro (set the time step back to 0.01). How is checking this result different from what we did in #7?

10. Acceleration due to gravity on Mars is 3.77 m/s^2. How does a projectile’s path on Mars differ from its path on Earth? *List THREE ways!*

11. What this macro does is generates a **model** of projectile motion. How is this different from solving projectile problems like we did in questions 1-3?

12. (a) Describe an experiment that you and your partner(s) could conduct to generate the same data that our macro produces.

    (b) Describe the limitations of this experiment.

    (c) Would you expect your experimental data to be similar to the data our macro produces? Explain why or why not.

Part Two: Lab Activity

Materials: Foam rocket, protractor, two meter sticks, stopwatch, graph paper

**NOTE: This portion of the lab must be completed outside

13. In the space below, derive the formula for calculating the range of a projectile.

14. You and your lab partner(s) are going to investigate the relationship between the range of a projectile and its initial launch angle. In the space below, write a procedure describing how to collect your data. Remember that you need a way to determine the initial velocity of your rocket!!

15. Take your procedure to the teacher, who will allow you to collect your materials and begin the lab.

16. In the space below, create a neat chart to organize all of your data. Make sure your chart is properly labeled and has appropriate units.

17. In the space below, calculate the initial velocity of your rocket. Make sure you circle your final answer.

18. Using your data, graph range vs. launch angle on the provided graph paper. Connect the dots.

19. Go back to the Excel file and open up the screen so that you can edit the macro. Go to **line 65**. You'll see that you need to input our range formula for calculating the range of the projectile. Type it in and exit back to the spreadsheet screen.

20. Make sure that the initial velocity on your spreadsheet is the initial velocity you calculated in #17. It doesn't matter what the angle is. Run the macro and select 'yes' when it asks you if you want to calculate the range data.

21. On your hand-drawn graph, plot the points from the Excel file that correspond to the angles you measured. If your experimental range matches the theoretical range for a particular angle, circle that data point. Connect these dots either with a different color pencil or a pen.
22. Attach your graph to this handout!

<u>Part Three: Conclusions</u>

23. Do you believe that this experiment was a good way to relate range and angle? Explain your answer in terms of how your experimental data compared to your theoretical data from the macro.

24. List at least three sources of error in your experiment.

-
-
-

25. Using your answers to #23 and #24, write a **formal** conclusion to this lab. Remember to write in complete sentences, state your purpose, explain your sources of error and whether they were small enough for your results to be valid. Additionally, discuss how this experiment relates to what we've discussed previously about the angles at which maximum ranges occur. Why did we create a model prior to conducting the experiment? Could we have developed our model *from* our experimental data?

Appendix B

Names: ____

Activity: Modeling Projectile Motion

Solve the following projectile problems with a partner on a separate piece of paper. Make sure you show your work.

1. A ball rolls off a 0.78-m high lab table and strikes the floor 0.55-m from the edge of the table. What was the initial velocity of the ball?

2. A toy rocket launcher has an initial launch speed of 2.44 m/s. If the launcher is angled at 30˚, how high in the air will the toy rocket go?

3. The launcher from the previous problem is placed on the 0.78-m high lab table and angled at 60˚. What is the horizontal range of the toy rocket at this angle?

Instead of using our formulas just to solve problems, let's think of them as *functions* instead.

4. A projectile has an initial velocity of v_0 and an initial launch angle of θ. Write expressions for both its horizontal AND vertical displacements as a function of time, assuming that the projectile is launched from an initial positive height of y_0 (consider the ground y = 0). *HINT: This isn't as difficult as it sounds – don't complicate it!*

Take your paper to the teacher to check. DO NOT MOVE ON UNTIL YOUR ANSWERS TO #4 HAVE BEEN INITIALED!

Now that you have your *x*- and - and *y*-functions, it's time to log on to your computer. Open the Excel file that the teacher has given you access to.

This Excel file uses a special program called a 'macro.' What a macro basically does is allows a user to write out all of the math in advance and run it over and over without having to retype anything.

Our macro does two things, but we're only going to use one of them for now. If there's a yellow bar across your screen telling you that the macro has been disabled, the first thing you need to do is click 'Enable' so that the program works properly.

Notice the top left-hand corner of the screen – it lists five variables:

- v_0 is the initial velocity.
- g is the acceleration due to gravity.
- θ is the initial angle.
- Δt is called the **time step** (basically, how often do you want the macro to calculate data points). Set it equal to 0.01, so that every second, it will calculate 100 times
- y_0 is the initial height.

Go to the "View" tab. On the far right of the menu bar, you should see a button that says "Macros." Click on it. A box pops up listing all the macros in this file (there's only one). Click "Edit."

The macro in this file is incomplete! It's up to you to enter the parameters so that it will calculate everything properly.

Go to **Line 30** – the formulas for *x* and *y* are missing! Using your equations from #4, fix the macro so that it will run.

When you're done, go to "File" and select "Save and Exit to Excel." You should be back to the spreadsheet. Click on "Macros" again, but now select "**Run**" in the box.

When it asks you if you want to calculate the range for all angles, select "No." We'll go back to that later.

5. Use this macro to check your answers to #1-3. In a couple of sentences, explain your method for checking these answers.

6. Look at the time step value – change it a few times (I suggest 1, 0.1, and 0.001 if you check nothing else). Is it possible for the time step to be too large? Too small? Explain your responses.

7. Why is *g* a variable? Explain how this could be useful to us.

8. What this macro does is generates a **model** of projectile motion. How does generating a model differ from solving problems? Why might we prefer a model to a problem?

9. (a) Describe an experiment that you and your partner(s) could conduct to generate the same data that our macro produces.
   (b) Describe the limitations of this experiment.
   (c) Would you expect your experimental data to be similar to the data our macro produces? Explain why or why not.

Names: ____

Lab: Determining the Range of a Projectile

Prelab Exercise

In your lab book, derive:

1. The expression for the range *R* of a projectile with initial velocity v_0 and initial angle θ when the projectile is launched from $y = 0$.
2. The expression for the time, t_{range}, that a projectile with initial velocity v_0 and initial angle θ when the projectile is launched from the height y_0.

The purpose of this lab is to compare the theoretical range calculation of a projectile to the experimental range measurements. You will be using a foam rocket as your projectile. You will additionally be given two meter sticks, a stopwatch, a protractor, and access to our Excel macro. Before collecting any experimental data, open the Excel macro back up and go down to **line 68** – you will need to enter the range formula we've discussed in the prelab exercise.

In your lab book, brainstorm your procedure with your partner before beginning. Make sure to develop a method for measuring/calculating the projectile's initial velocity as well as the range. Also remember that the projectile is not launched from $y = 0$, so you will need to account for that as well.

Organize your data neatly, and make sure to show all work in calculating the initial velocity. Once you have done this, you will use Excel to generate a graph with two lines: your experimental range data as a function of angle **and** the theoretical range data given by the Excel macro (with your correctly measured initial velocity) on the same plot. Use the 'scatter plot with smooth lines AND data points' on your experimental data and the smooth line plot on your theoretical data.

You will be writing a formal lab report to conclude this exercise. It will include the following components:

TITLE: Your lab report title is the same title as the one above ☺

ABSTRACT: This is 80-250 words summarizing your experiment. It should state our purpose and briefly explain how you met it. Think of it as 'hitting the high points.'

INTRODUCTION: Go over all of the math theory we've discussed regarding projectiles. You should also present your derivation of the range formula here. Use parenthetical citations when appropriate.

METHODOLOGY: Your procedure. In detail. Use diagrams when appropriate.

DATA AND RESULTS: Present your graph only here – no charts. Show your calculation for the initial velocity; make sure to explain in words why you're doing the math you're showing. Based on your experimental data, determine the angle corresponding to the maximum range (explain your method for doing this). Find this angle's percent difference from 45˚.

CONCLUSION: Discuss whether or not you met your purpose, what your sources of error were (be detailed). Comment on the macro we used to generate our data: what were its benefits and limitations. Finally, discuss your maximum range angle and why it should/shouldn't be the ideal 45˚.

REFERENCES: Any references should be cited in MLA style.