INSTRUCTIONS

1. Please check to ensure that you have a complete exam booklet. There are 22 numbered problems. Note that Problem 2 occupies 2 pages, Problem 3 occupies 2 pages, Problem 10 occupies 2 pages, Problem 21 occupies 6 pages. Including the cover sheet, you should have 60 pages. There should be no blank pages in the booklet.

2. The examination is closed book and closed notes. No reference material is allowed at your desk. A calculator is permitted.

3. All wireless devices must be turned off for the entire duration of the exam.

4. You may work a problem directly on the problem statement (if there is room) or on blank sheets of paper available from the exam proctor. Do not write on the back side of any sheet.

5. Your examination code number MUST APPEAR ON EVERY SHEET. This includes this cover sheet, the problem statement sheets, and any additional work sheets you turn in. DO NOT write your name on any of these sheets. Use the preprinted numbers whenever possible, or WRITE LEGIBLY!!

6. Under the rules of the examination, you must choose 8 problems to be handed in for grading. Each problem to be graded should be separated from the rest of the materials, stapled to the associated worksheets, and placed on the top of the appropriate envelope in the front of the exam room. DO NOT TURN IN ANY SHEETS FOR THE OTHER 17 PROBLEMS!!

7. The examination lasts 4 hours, from 9:30 AM to 1:30 PM EST.

8. When you hand in the exam:

   (a) Separate the 8 problems to be graded as explained above.

   (b) Check to see that your Code Number is in EVERY sheet you are turning in.

   (c) On the section at the bottom of this page, CIRCLE the problem numbers that you are turning in for grading.

   (d) Turn in this cover sheet (containing your code number) and the 8 problems to be graded.

   (e) All other material is to be placed in the discard box at the front of the room. You are not allowed to take any of the exam booklet pages from the room!

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Problem 1 (Core: VSDD-ECE2020)  

Prelim Solutions

PROBLEM

Answer all the parts of the question.

a. (3 pts) The following truth table describes the behavior of an encoder with an unusual priority. Based on the don't cares of the inputs, list the inputs from the highest priority to lowest.

<table>
<thead>
<tr>
<th>$IN_3$</th>
<th>$IN_2$</th>
<th>$IN_1$</th>
<th>$IN_0$</th>
<th>$OUT_1$</th>
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b. (5 points) Consider the circuits below. Here the nodes Q1, Q2, Q3 and Q4 are being observed. The timing diagrams for CLK and DATA are also given. Draw the timing diagram for the signals Q1, Q2, Q3 and Q4 in the space provided. Assume that the signals Q1, Q2 and Q3 are initially 0.

![Circuit Diagram](image)

![Timing Diagram](image)

c. (2 points) Use 2's complement representation to perform the following computation. Use the minimum number of bits required to perform the computation.

i) $-13-27$
a.  IN2>IN0>IN3>IN1

b.  

\[ \text{CLK} \]

\[ \text{DATA} \]

\[ \text{Q1} \]

\[ \text{Q2} \]

\[ \text{Q3} \]

\[ \text{Q4} \]

c.  Minimum no of bits required is 7.
\(-13\)\(_{10}\): (1110011)\(_2\)
\(-27\)\(_{10}\): (1100101)\(_2\)
Adding these two numbers we get (1011000)\(_2\) which is (-40)\(_{10}\)
Problem 2 (Core: DSP-ECE2026)  Prelim Solutions

**Problem**

\[ \begin{array}{c}
\text{IDEAL} \\
\text{C-to-D} \\
\text{CONVERTER} \\
\hspace{1cm} f_s \\
\text{IDEAL} \\
\text{D-to-C} \\
\text{CONVERTER} \\
\hspace{1cm} f_s \\
x(t) \rightarrow \text{x}[n] \rightarrow \text{h}[n] \rightarrow \text{y}[n] \rightarrow \text{y}(t)
\end{array} \]

The input to the above system is defined by

\[ x(t) = \sum_{k=-3}^{3} a_k e^{j100\pi kt}, \text{ where } a_k = \begin{cases} 
\frac{1}{2\pi(4 + k^2)}, & k \neq 0 \\
\frac{1}{2}, & k = 0
\end{cases} \]

Assume that \( f_s = 400 \text{ Hz} \)

(a) Is \( x(t) \) periodic? If so, find the period \( T_0 \) in seconds. If not, explain why it is not periodic.

If periodic: \( T_0 = \)

If not periodic (explain):

(b) Plot the Discrete-Time Fourier Transform (DTFT) for \( x[n] \) (i.e., find \( X(e^{j\omega}) \))

(Label all complex amplitudes and the \( \omega \)-axis).
Problem 2 (Core: DSP-ECE2026) Prelim Solutions

(c) The LTI system $h[n]$ is defined as:

$$h[n] = \delta[n - 1] + 2\delta[n - 4] + \delta[n - 9]$$

Find an expression for the magnitude and phase of the Discrete-Time Fourier Transform (DTFT) for $h[n]$ (i.e., find $|H(e^{j\omega})| e^{j\angle H(e^{j\omega})}$).

\[
\begin{align*}
|H(e^{j\omega})| &= \\
\angle H(e^{j\omega}) &=
\end{align*}
\]

(d) For $x(t)$ given above, the output signal can be written as

$$y(t) = \sum_{k=0}^{3} B_k \cos(\omega_0 kt + \phi_k)$$

Determine the numerical values of the parameters:

$B_0, (B_1, \phi_1), (B_2, \phi_2), (B_3, \phi_3),$ and $\omega_0$.

\[
\begin{align*}
B_0 &= \\
(B_2, \phi_2) &= \\
(B_1, \phi_1) &= \\
(B_3, \phi_3) &= \\
\omega_0 &=
\end{align*}
\]
Problem 2 (Core: DSP-ECE2026)  Prelim Solutions

SOLUTION

(a) Is \( x(t) \) periodic? If so, find the period \( T_0 \) in seconds. If not, explain why it is not periodic.

\[
x(t) = \sum_{k=-3}^{3} a_k e^{j100\pi kt}, \text{ where } a_k = \begin{cases} 
1 & k \neq 0 \\
\frac{1}{2} & k = 0 
\end{cases}
\]

\( x(t) \) is written as a Fourier Synthesis equation which only applies to periodic signals. The complex exponentials are all harmonics of a fundamental frequency: \( f_0 = 50 \, \text{Hz} \). Therefore, the signal is periodic with:

\[
T_0 = \frac{1}{50} = 0.02 \, \text{secs}
\]

(b) Plot the Discrete-Time Fourier Transform (DTFT) for \( x[n] \) (i.e., find \( X(e^{j\hat{\omega}}) \)).

\[
a_k = \begin{cases} 
\frac{1}{2\pi(4 + k^2)} & k \neq 0 \\
\frac{1}{2} & k = 0 
\end{cases}
\]

\[
a_0 = \frac{1}{2}, a_{\pm 1} = \frac{1}{10\pi}, a_{\pm 2} = \frac{1}{16\pi}, a_{\pm 3} = \frac{1}{26\pi}
\]

\[
\hat{\omega} = \frac{2\pi f}{f_s}
\]

\( f_s = 400; \hat{\omega}_0 = 0, \hat{\omega}_{\pm 1} = \pm \frac{\pi}{4}, \hat{\omega}_{\pm 2} = \pm \frac{\pi}{2}, \hat{\omega}_{\pm 3} = \pm \frac{3\pi}{4}
\]
(c) \( h[n] = \delta[n - 1] + 2\delta[n - 5] + \delta[n - 9] \)

Find an expression for the magnitude and phase of the Discrete-Time Fourier Transform (DTFT) for \( h[n] \) (i.e., find \( |H(e^{j\hat{\omega}})| \) and \( \angle H(e^{j\hat{\omega}}) \)).

\[
H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}}(2 + 2e^{-j4\hat{\omega}} + e^{-j9\hat{\omega}})
\]

\[
H(e^{j\hat{\omega}}) = (2 + 2\cos(4\hat{\omega}))e^{-j5\hat{\omega}}
\]

| \( |H(e^{j\hat{\omega}})| = 2 + 2\cos(4\hat{\omega}) \) | \( \angle H(e^{j\hat{\omega}}) = -5\hat{\omega} \) |

(d) For \( x(t) \) given above, the output signal can be written as

\[
y(t) = \sum_{k=0}^{3} B_k \cos(\omega_0 kt + \phi_k)
\]

Determine the numerical values of the parameters:

\( B_0, (B_1, \phi_1), (B_2, \phi_2), (B_3, \phi_3), \) and \( \omega_0 \).

From (b) we know that, with \( f_s = 400, \) \( x[n] \) has digital frequency content at:

\( \hat{\omega}_0 = 0, \hat{\omega}_{\pm 1} = \pm \frac{\pi}{4}, \hat{\omega}_{\pm 2} = \pm \frac{\pi}{2}, \hat{\omega}_{\pm 3} = \pm \frac{3\pi}{4} \)

To find \( y(t) \), it is necessary to determine what happens to the content at these frequencies when passed through the digital filter with frequency response: \( H(e^{j\hat{\omega}}) = (2 + 2\cos(4\hat{\omega}))e^{-j5\hat{\omega}} \).

\[
H(e^{j\hat{\omega}}) = 0, \text{for} \hat{\omega} = \pm \frac{\pi}{4}, \pm \frac{3\pi}{4}
\]

\[
H(e^{j\hat{\omega}}) = 2, \text{for} \hat{\omega} = 0
\]

\[
H(e^{j\hat{\omega}}) = 4e^{\mp j\frac{3\pi}{2}} = 4e^{\mp j\frac{\pi}{2}}, \text{for} \hat{\omega} = \pm \frac{\pi}{2}
\]

Therefore, if we pass \( x(t) \) through the digital filter, the remaining values will be:

\( B_0 = a_0 * 2 = 1; \check{B}_{\pm 1} = a_{\pm 1} * 0 = 0; \check{B}_{\pm 2} = a_{\pm 2} * 4e^{\mp j\pi/2} = \frac{1}{4\pi}e^{\mp j\pi/2}; \check{B}_{\pm 3} = a_{\pm 3} * 0 = 0 \)
There is no aliasing so $\omega_0$ remains $100\pi$.

The final answer is:

$$y(t) = 1 + \frac{1}{2\pi} \cos\left(200\pi t - \frac{\pi}{2}\right)$$

\[B_0 = 1\]

\[(B_2, \phi_2) = \left(\frac{1}{4\pi}, -\frac{\pi}{2}\right)\]

\[(B_1, \phi_1) = 0\]

\[(B_3, \phi_3) = 0\]

$\omega_0 = 100\pi$
The attached C++ program compiles and executes without errors. In the table below, fill in the output lines produced by this C++ program in the table below. There may be extra spaces in the table.

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// Program Code for ECE 2036 C++ Problem
#include <iostream>
using namespace std;

class test
{
public:
    test(int x);
    ~test()
    {
        cout << "Dest" << endl;
    }
    void x(test &y);
    int w;
};
test::test(int x)
: w(x+1)
{
    cout << "Cons" << endl;
}
void test::x(test &y)
{
    cout << y.w << endl;
    if (y.w>0) y.w = -y.w;
}

int main()
{
    test A(1);
    test B(0);
    A.x(A);
    B = A;
    cout << B.w << endl;
    int a[9]={0,1,2,3,4,5,6,7,8};
    int *aptr;
    aptr = &a[0];
    ++aptr;
    (*aptr)--;
    (*(++aptr))++;
}
SOLUTION

```plaintext
Cons
Cons
2
-2
0
4
7
4
Dest
Dest
Press any key to continue . . . .
```

11 of 60
Analyze a circuit shown below.

PART I: Assume that \( v_s(t) = 30 - 40e^{-5t}u(t) \).

(i) (2 points) Find the time constant of the natural response of \( i(t) \) in the units of second.

(ii) (4 points) Find the complete response for \( i(t) \).

PART II: Assume \( v_s(t) = 20\cos(\omega t) \) and the system is in the steady state.

(iii) (3 points) Find the network function \( H(\omega) \), defined by \( H(\omega) = \frac{V_o(\omega)}{V_s(\omega)} \).

(iv) (1 point) Find the corner frequency(-ies) of the network function.

No credits will be given for answers without the step-by-step work that is legibly written on the answer sheets.
(i) First, find the Norton equivalent circuit that excludes the inductor network. One has \( V_{oc}(t) = 0.4V_t(t) \), \( R_s = 60 \Omega \), \( i_{sc}(t) = v_{s}(t)/150 \).

Then, find the equivalent inductance (\( L_{eq} \)): \( L_{eq} = 5H/20 = 4 \) H.

The equivalent circuit is:

\[
\begin{align*}
\text{The time constant of the natural response (\( \tau \)) can be calculated as:} \\
\tau &= \frac{L_{eq}}{R_t} = \frac{4}{60} = \frac{1}{15} = 0.067 \text{ s} \\
\end{align*}
\]

(ii) The first-order circuit equation is:

\[
\frac{di_c}{dt} + \frac{R}{L} \cdot i_c + i = \frac{R}{L} \cdot i_f \quad \Rightarrow \quad \frac{di_c}{dt} + 15i_c = 3 - 4e^{-5t}u(t)
\]

The general solution to the natural response \( i_n \) and the forced response \( i_f \) can be expressed as:

\[ i_n = Ke^{-15t}; \quad i_f = C + Ae^{-5t} \Rightarrow i(t) = Ke^{-15t} + C + Ae^{-5t} \]

Applying the boundary conditions: \( i(0) = 0.2 \) (A); \( i(\infty) = 0.2 \) (A)

First, solve for the constants of the forced response for \( t > 0 \):

Use the differential equation:

\[ \frac{di_f}{dt} + 15i_f = -5Ae^{-5t} + 15C + 15Ae^{-5t} = 3 - 4e^{-5t} \]

Compare the coefficients, one has \( C = 0.2 \) & \( A = -0.4 \).

Then, use the initial condition:

\[ i(0) = Ke^{-15(0)} + 0.2 - 0.4e^{-5(0)} = 0.2 \Rightarrow K = 0.4. \]

The complete response is:

\[ i(t) = 0.4e^{-15t} + 0.2 - 0.4e^{-5t} \]

(iii) Use the phasor expression: \( V_s(t) \rightarrow V_s(\omega) \)

The Thévenin equivalent circuit derived from part (i) is:

\[
\begin{align*}
\text{The network function can be found by:} \\
V_s(\omega) &= 0.4V_s(\omega) + \frac{j\omega L_{eq}}{R_s + j\omega L_{eq}} = \frac{0.4}{1 + \frac{R_s}{j\omega L_{eq}}}V_s(\omega) = \frac{0.4}{1 - j\left(\frac{R_s}{L_{eq}}\right)}V_s(\omega) \\

\text{Let } \omega_0 &= \frac{R_s}{L_{eq}} = 15 \text{ rad/s} \\
H(\omega) &= \frac{V_s(\omega)}{V_s(\omega)} = \frac{0.4}{1 - j\left(\frac{15}{\omega}\right)} = \frac{0.4}{1 - j\left(\frac{15}{\omega}\right)} \\
\text{The corner frequency is } \omega_0 = 15 \text{ rad/s}
\end{align*}
\]
Problem 5 (Core: CSS-ECE3020) Prelim Solutions

PROBLEM

A builder is constructing four houses on a street, as shown in the diagram to the right. Each house has two “adjacent” houses and one “opposite” house. (For example, houses B and D are adjacent to house A and house C is opposite house A.)

Assume there are six distinct possible house colors, one of which is green.

Calculate each of the following values. Clearly label each part and draw a box around your final numerical answer for each question. Additionally, to receive full credit, you MUST provide appropriate equations/calculations and a brief description of the rationale or logic for each answer.

For parts (A) – (D), assume that each house may be painted any of the six colors.

(A) 1 point – How many distinct arrangements of houses and colors are possible?

(B) 1 point – How many arrangements have each house painted a different color?

(C) 2 points – How many arrangements have at most two houses painted in each color?

(D) 3 points – Assume all color choices are equally likely. What is the probability that exactly two houses are painted green and are located adjacent to each other? What is the probability that exactly two houses are painted green and are located opposite of each other?

(E) 3 points – Assume that no house may be painted the same color as either of the adjacent houses and that all allowable color choices are equally likely. What is the probability that a house is painted the same color as the house located opposite it?
Problem 5 (Core: CSS-ECE3020)  Prelim Solutions

SOLUTION

A builder is constructing four houses on a street, as shown in the diagram to the right. Each house has two “adjacent” houses and one “opposite” house. (For example, houses B and D are adjacent to house A and house C is opposite house A.)

Assume there are six distinct possible house colors, one of which is green.

Calculate each of the following values. Clearly label each part and draw a box around your final numerical answer for each question. Additionally, to receive full credit, you MUST provide appropriate equations/calculations and a brief description of the rationale or logic for each answer.

For parts (A) – (D), assume that each house may be painted any of the six colors.

(A) 1 point – How many distinct arrangements of houses and colors are possible?

6 choices for first house, 6 choices for second, etc.

\[(6) (6) (6) (6) = 6^4 = 1296\]

(B) 1 point – How many arrangements have each house painted a different color?

6 choices for first house, 5 choices for second, 4 choices for third, etc.

\[(6) (5) (4) (3) = 6! / 2! = 360\]

(C) 2 points – How many arrangements have at most two houses painted in each color?

# max 2 = # possible − # with 3 same − # with 4 same

# 3 same: xxyy or xyyx or xyxx or yxxx; 6 choices for x, 5 choices for y

=> \[(4) (6) (5) = 120\]

# 4 same: 6 choices of color => 6

# max 2 = 1296 − 120 − 6 = 1170
Problem 5 (Core: CSS-ECE3020)  Prelim Solutions

(E) 3 points – Assume all color choices are equally likely. What is the probability that exactly two houses are painted green and are located adjacent to each other? What is the probability that exactly two houses are painted green and are located opposite of each other?

\[
\text{Prob (2 green, adj)} = \frac{\text{# (2 green, adj)}}{\text{# possible}}
\]
\[
\text{# (2 green, adj): } \text{GGxy or xGGy or xyGG or GxyG; 5 choices for x, y } \Rightarrow (4) (5) (5) = 100
\]
\[
\text{Prob (2 green, adj)} = \frac{100}{1296} \approx 0.0772 \quad 0.0772 = 7.72\%
\]
\[
\text{Prob (2 green, opp)} = \frac{\text{# (2 green, opp)}}{\text{# possible}}
\]
\[
\text{# (2 green, opp): } \text{GxGy or xGyG; 5 choices for x, y } \Rightarrow (2) (5) (5) = 50
\]
\[
\text{Prob (2 green, opp)} = \frac{50}{1296} \approx 0.0386 \quad 0.0386 = 3.86\%
\]

(F) 3 points – Assume that no house may be painted the same color as either of the adjacent houses and that all allowable color choices are equally likely. What is the probability that a house is painted the same color as the house located opposite it?

Since all houses are symmetric, arbitrarily choose \(\text{Prob (colorA = colorC)}\).

There are six choices for colorA; colorB and colorD cannot be the same as colorA, so they have the same set of five choices. ColorC cannot be the same as either colorB or colorD. If colorB is the same as colorD, then there are five choices for colorC; if colorB is not the same as colorD, then there are four choices for colorC. In either case, colorA is one of the choices available for colorC.

\[
\text{Prob (colorA = colorC)} = \text{Prob (colorA = colorC} \mid \text{colorB = colorD)}
\]
\[
+ \text{Prob (colorA = colorC} \mid \text{colorB \neq colorD)}
\]
\[
\text{Prob (colorA = colorC)} = \text{Prob (cA = cC} \mid \text{cB = cD)} \cdot \text{Prob (cB = cD)}
\]
\[
+ \text{Prob (cA = cC} \mid \text{cB \neq cD)} \cdot \text{Prob (cB \neq cD)}
\]
\[
\text{Prob (colorA = colorC)} = \left(\frac{1}{5}\right) \cdot \left(\frac{1}{5}\right) + \left(\frac{1}{4}\right) \cdot \left(\frac{4}{5}\right)
\]
\[
= \left(\frac{.2}{.2}\right) + (.25) (.8)
\]
\[
= .04 + .20 = .24 \quad 0.24 = 24\%
\]
A time domain reflectometer (TDR) is connected to a transmission line system as shown below. The voltage $V$ is recorded by the TDR with the source pulse shown below. Determine the transit times $\tau_1$ and $\tau_2$ and the impedances $Z_1$, $Z_2$, and $Z_3$. 

![Diagram of a time domain reflectometer connected to a transmission line system with voltages and time scales.]
A time domain reflectometer (TDR) is connected to a transmission line system as shown below. The voltage $V$ is recorded by the TDR with the source pulse shown below. Determine the transit times $\tau_1$ and $\tau_2$ and the impedances $Z_1$, $Z_2$, and $Z_3$.

\[
\begin{align*}
A: \quad V &= 75V \rightarrow Z_1 = 56\Omega \\
B: \quad \tau_1 &= 3\mu s, \quad \tau_2 = 2.5\mu s, \quad Z_2 = 100\Omega \\
\Delta V &= -15V \rightarrow \tau_1 = \frac{75}{8} = 9.375 \mu s \rightarrow Z_1 = \frac{Z_1}{Z_2} = \frac{1}{1+\tau_1} = 0.125 \\
C: \quad \text{End of 75V pulse} \\
D: \quad \tau_2 &= 2\mu s, \quad \tau_3 = 2\mu s \\
\Delta V &= -15V \rightarrow \tau_2 = (1+\tau_1)(1+\tau_2) \rightarrow \tau_2 = -\frac{1}{3} \\
\tau_3 &= \frac{Z_2(Z_1-Z_2)}{Z_1(Z_1-Z_2)} = -\frac{3}{5} \rightarrow Z_3 = \frac{Z_2}{1+\tau_3} = 50\Omega \\
\tau_1 &= 1.5\mu s \\
\tau_2 &= 2\mu s \\
Z_1 &= 56\Omega \\
Z_2 &= 100\Omega \\
Z_3 &= 50\Omega
\end{align*}
\]
Assume that you have to budget for power needed for the following memory unit to operate properly: The cache operates at 1 GHz, the supply voltage is $V_{dd} = 2V$ and L1 Instruction Cache has capacitance of 1.1 pF and activity factor of 0.9, L1 Data Cache has capacitance of 0.6 pF and activity factor of 0.3, L2 cache has $C=20$ nF and activity factor of 0.05.

(a) What is the power consumption of this system?

(b) One DRAM cell in L1 Cache has a storage capacitance equal to 1.5 fF. Its bit line is a poly-wire of size 0.5μm $\times$ 250μm with length of 1mm and the copper resistivity is $15\times10^{-9}$ Ωm. What is the 50% delay of this DRAM cell?
Assume that you have to budget for power needed for the following memory unit to operate properly: The cache operates at 1 GHz, the supply voltage is $V_{dd} = 2V$ and L1 Instruction Cache has capacitance of 1.1 pF and activity factor of 0.9, L1 Data Cache has capacitance of 0.6 pF and activity factor of 0.3, L2 cache has $C=20$ nF and activity factor of 0.05.

(a) What is the power consumption of this system?

$$R_1 = \frac{1}{2} V_{dd}^2 C_{L1} f \alpha_1 = 0.5 \cdot 2^2 \cdot 1.1 \cdot 10^{-12} \cdot 10^9 \cdot 0.9 = 0.00198 \text{ W}$$

$$P_2 = \frac{1}{2} V_{dd}^2 C_{L1D} f \alpha_2 = 0.5 \cdot 2^2 \cdot 0.6 \cdot 10^{-12} \cdot 10^9 \cdot 0.3 = 0.00036 \text{ W}$$

$$P_3 = \frac{1}{2} V_{dd}^2 C_{L2} f \alpha_3 = 0.5 \cdot 2^2 \cdot 20 \cdot 10^{-9} \cdot 10^9 \cdot 0.05 = 2 \text{ W}$$

$$P = R_1 + P_2 + P_3 = 2.002 \text{ W}$$

(b) One DRAM cell in L1 Cache has a storage capacitance equal to 1.5 fF. Its bit line is a poly-wire of size $0.5 \mu m \times 250 \mu m$ with length of 1mm and the copper resistivity is $15 \times 10^{-9} \Omega m$. What is the 50% delay of this DRAM cell?

$$R = \frac{\rho \cdot L}{A} = \frac{15 \cdot 10^{-9} \cdot 0.001}{0.5 \cdot 10^{-6} \cdot 250 \cdot 10^{-6}} = 0.12 \Omega$$

$$\tau_{50\%} = 0.693RC = 0.693 \cdot 0.12 \cdot 1.5 \cdot 10^{-15} = 1.247 \cdot 10^{-16} s$$
The following parameters of Si at 300 K may be useful:
$E_g = 1.12 \, \text{eV}; \quad m_e^* = 1.09 \, m_0; \quad m_h^* = 1.15 \, m_0; \quad n_i = 1 \times 10^6 \, \text{cm}^{-3}; \quad N_C = 2.86 \times 10^{19} \, \text{cm}^{-3};$
$N_V = 3.10 \times 10^{19} \, \text{cm}^{-3}; \quad \varepsilon_r = 11.8; \quad k = 8.62 \times 5 \, \text{eV} / \text{K}$

The band-edge diagram along the vertical (x) direction of a Si n-channel MOSFET at 300 K is described by:
$E_C(x) = 1.02 \, \text{eV}, \quad x < 0 \, \text{cm}$
$E_C(x) = 1.02 - (0.9e8)x^2 \, \text{eV}, \quad 0 \, \text{cm} < x < 1 \times 10^{-4} \, \text{cm}$
$E_F(x) = 0 \, \text{eV}, \quad \text{for all } x$
where $x$ is in cm. This band edge diagram is sketched below.

\[ E_C \quad x = 0 \, \text{cm} \quad x = 1 \times 10^{-4} \, \text{cm} \]

(a) Calculate $n$ and $\rho$ in $\text{cm}^{-3}$ at $x = 0 \, \text{cm}$.

(b) Suggest possible values for the donor and acceptor densities $N_A$ and $N_D$ at $x = 0 \, \text{cm}$. There is more than one set of possible values, but you are only expected to provide one set.

(c) Derive an expression for the electric field $\mathcal{E}(x)$ for $x < 0 \, \text{cm}$ and $0 \, \text{cm} < x < 1 \times 10^{-4} \, \text{cm}$.

(d) Answer each of the following with one of these three: “left”, “right”, or “zero magnitude”.
   (i) Direction of the diffusion current at $x = 5 \times 10^{-5} \, \text{cm}$:
   (ii) Direction of the drift current at $x = 5 \times 10^{-5} \, \text{cm}$:
   (iii) Direction of the total current at $x = 5 \times 10^{-5} \, \text{cm}$:
Problem 8 (Core: Microsystems-ECE3040)  Prelim Solutions

**SOLUTION**

The following parameters of Si at 300 K may be useful:

- \( E_g = 1.12 \text{ eV} \)
- \( m^*_e = 1.09 \text{ m}_0 \)
- \( m^*_h = 1.15 \text{ m}_0 \)
- \( n_i = 1 \times 10^{10} \text{ cm}^{-3} \)
- \( N_C = 2.86 \times 10^{19} \text{ cm}^{-3} \)
- \( N_V = 3.10 \times 10^{19} \text{ cm}^{-3} \)
- \( \varepsilon_s = 11.8 \)
- \( k = 8.62 \times 10^{-5} \text{ eV/K} \)

The band-edge diagram along the vertical (x) direction of a Si n-channel MOSFET at 300 K is described by:

- \( E_C(x) = 1.02 \text{ eV}, \quad x < 0 \text{ cm} \)
- \( E_C(x) = 1.02 - (0.9e8)x^2 \text{ eV}, \quad 0 \text{ cm} < x < 1 \times 10^{-4} \text{ cm} \)
- \( E_F(x) = 0 \text{ eV}, \quad \text{for all } x \)

where \( x \) is in cm. This band edge diagram is sketched below.

![Band Edge Diagram](image)

(a) Calculate \( n \) and \( p \) in \( \text{cm}^{-3} \) at \( x = 0 \text{ cm} \).

\[
\begin{align*}
    p &= N_V \exp \left( \frac{E_V - E_F}{kT} \right) = N_V \exp \left( \frac{-1.12 \text{ eV}}{-11.8 \text{ eV} / \text{K}} \right) \\
    &= 6.52 \times 10^{17} \text{ cm}^{-3} \\
    n &= n_i^2 / p = 1.53 \times 10^{14} \text{ cm}^{-3}
\end{align*}
\]

(b) Suggest possible values for the donor and acceptor densities \( N_A \) and \( N_D \) at \( x = 0 \text{ cm} \).

There is more than one set of possible values, but you are only expected to provide one set.

\[
\begin{align*}
    N_A &= 6.52 \times 10^{17} \text{ cm}^{-3} \\
    N_D &= 0 \text{ cm}^{-3}
\end{align*}
\]

(c) Derive an expression for the electric field \( E(x) \) for \( x < 0 \text{ cm} \) and \( 0 \text{ cm} < x < 1 \times 10^{-4} \text{ cm} \).

\[
\begin{align*}
    x < 0 \text{ cm} \quad E(x) &= 0 \\
    0 < x < 1 \times 10^{-4} \text{ cm} \quad E(x) &= \frac{1}{\varepsilon_s} \frac{dE_F}{dx} \\
        &= -1.8 \times 10^8 \times \varepsilon \text{ V/cm}
\end{align*}
\]

(d) Answer each of the following with one of these three: “left”, “right”, or “zero magnitude”.

(i) Direction of the diffusion current at \( x = 5 \times 10^{-5} \text{ cm} \): right

(ii) Direction of the drift current at \( x = 5 \times 10^{-5} \text{ cm} \): left

(iii) Direction of the total current at \( x = 5 \times 10^{-5} \text{ cm} \): 0 magnitude
The minimum incident solar energy at the location of the PV system is 5 kWh/m²/day. A 4 kW PV system is to be designed with 16 percent efficient panels rated 200 W each and installed on the roof of the house. The load is to be served by the electrochemical batteries which are charged to full capacity during the day (with charge/discharge cycle efficiency of 80 percent). The batteries are allowed to discharge 80 percent of their full capacity while feeding the load. Calculate the power of the night-time load which needs to be supplied from the batteries for 6 hours each night. NOTE: please assume that the battery capacity is chosen so as to receive the full daily energy produced by the system.
Number of panels \( N = \frac{4kW}{200W} = 20 \)

Area of each panel \( A = \frac{200}{1000 \times 16\%} = 1.25 \ m^2 \)

Area of the system \( A_{sys} = 20 \times 1.25 \ m^2 = 25 \ m^2 \)

Daily energy yield \( W_d = 5kWh / m^2 / day \times 25 \ m^2 \times 16\% = 20kWh \)

Daily useful energy yield \( W_{daily} = 20kWh \times 80\% = 16 kWh \)

As allowed depth of discharge is 80%, only 16kWh \times 80\% = 12.8kWh will be provided to the load at the worst case.

The power of the night-time load \( P_{Load} = \frac{12.8kWh}{6h} = 2.13kW \)
In this problem, $X$ and $Y$ are independent and identically distributed random variables that are uniformly distributed on the interval $[0, 1]$. The probability density functions (pdfs) for $X$ and $Y$ are

$$f_X(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}, \quad f_Y(y) = \begin{cases} 1, & 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}.$$

a. Calculate $P(X^2 \leq Y)$, the probability that $X^2$ is less than or equal to $Y$.

b. Let $Z$ be another random variable which is the maximum of $X$ and $Y$, $Z = \max(X, Y)$. Find the probability density function for $Z$. 

c. Now set $Z = X^2$. Find the probability density function of $Z$. 
In this problem, $X$ and $Y$ are independent and identically distributed random variables that are uniformly distributed on the interval $[0, 1]$. The probability density functions (pdfs) for $X$ and $Y$ are

$$f_X(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}, \quad f_Y(y) = \begin{cases} 1, & 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}.$$

a. Calculate $P(X^2 \leq Y)$, the probability that $X^2$ is less than or equal to $Y$.

We integrate the joint density function (which is a constant over the unit square in this case) over the region $x^2 \leq y$

$$P(X^2 \leq Y) = \int_{x=0}^{1} \int_{y=x^2}^{1} 1 \, dy \, dx$$

$$= \int_{x=0}^{1} (1 - x^2) \, dx$$

$$= 1 - \frac{1}{3} = \frac{2}{3}.$$

b. Let $Z$ be another random variable which is the maximum of $X$ and $Y$, $Z = \max(X, Y)$. Find the probability density function for $Z$.

We derive the cumulative distribution function $F_Z(z) = P(Z \leq z)$ and then differentiate to get the pdf. It is clear that $F_Z(z) = 0$ for $z < 0$ and $F_Z(z) = 1$ for $z > 1$. For $0 \leq z \leq 1$, we have

$$F_Z(z) = P(Z \leq z) = P(X \leq z) \cdot P(Y \leq z)$$

$$= z^2,$$

and so the pdf is

$$f_Z(z) = 2z, \quad 0 \leq z \leq 1.$$
c. Now set $Z = X^2$. Find the probability density function of $Z$.

In this case, we have (for $0 \leq z \leq 1$),

\[
F_Z(z) = P(\{Z \leq z\}) = P(\{X^2 \leq z\}) = P(\{X \leq \sqrt{z}\}) = \sqrt{z},
\]

and so

\[
f_Z(z) = \frac{1}{2\sqrt{z}}, \quad 0 \leq z \leq 1.
\]
Problem 11 (Core: S&C-ECE3084)  Prelim Solutions

**Problem**

**Part a**

Given the function

\[ f_n(t) = n^2 \left[ u\left(t + \frac{1}{n}\right) - u(t)\right] - n^2 \left[ u(t) - u\left(t - \frac{1}{n}\right)\right] \]

Plot the convolution \( g_n(t) = f_n(t) \ast u(t) \). What is the limit of \( g_n \) as \( n \to \infty \)?

**Part b**

Consider the functions \( f_n(t) = \delta(t) \sin(t)^n, n = 1, 2, \ldots \). Plot the second derivative of \( f_n(t) \) for \( n = 1, 2, \ldots \).
Solution a

\[ g_n(t) = \begin{cases} 
  t \leq -\frac{1}{n}, & 0 \\
  -\frac{1}{n} < t \leq 0, & f_0^{t+\frac{1}{n}} n^2 dt = n^2 \left( t + \frac{1}{n} \right) \\
  0 < t \leq \frac{1}{n}, & f_0^t (-n^2) dt + f_t^{t+\frac{1}{n}} n^2 dt = n^2 \left( -t + \frac{1}{n} \right) \\
  t > \frac{1}{n}, & f_t^{t-\frac{1}{n}} (-n^2) dt + f_t^{t+\frac{1}{n}} n^2 dt = 0 
\end{cases} \]

The function \( g_n(t) \) has a triangular shape within \([-1/n, 1/n]\) with peak at \( t = 0 \) and zero outside. As \( n \to \infty \), \( g_n(t) \) tends to a Dirac delta. Indeed, \( \int g_n(t) dt = \frac{1}{2} (n)(\frac{2}{n}) = 1 \) for any \( n \) and the limiting function is zero everywhere, except at \( t = 0 \) where is infinitely large.

Solution b

Since \( f(t) = \delta(t) \sin(t)^n = 0 \delta(t) = 0 \) for \( n = 1, 2, ..., \) the derivative of any order is zero.
1. Design a 2-to-4 decoder using inverters, NANDs with up to four inputs, and NORs with up to four inputs. Your goal is to use as few gates as possible. Assume that inputs are available in both complemented and uncomplemented forms. Recall that a 2-to-4 takes in two bits and outputs four bits with a one-hot encoding, e.g., input 00 results in output 0001 while input 01 results in output 0010. Clearly label all inputs and outputs. In your design, use for NAND, for NOR and for an inverter.

2. Assuming that the mobility of electrons is four times as high as that of holes, design in CMOS a 4-input NAND gate with equal rise and fall times. Use symbol $w$ to indicate the width of a minimum-size nFET, use symbol $\uparrow$ for a pFET and use symbol $\downarrow$ to designate an nFET.
1. Design a 2-to-4 decoder using inverters, NANDs with up to four inputs, and NORs with up to four inputs. Your goal is to use as few gates as possible. Assume that inputs are available in both complemented and uncomplemented forms. Recall that a 2-to-4 takes in two bits and outputs four bits with a one-hot encoding, e.g., input 00 results in output 0001 while input 01 results in output 0010. Clearly label all inputs and outputs. In your design, use \( \nabla \) for NAND, for NOR and \( \nabla \) for an inverter.

Inputs are \( a_1, a_0 \), outputs \( d_3, d_2, d_1, d_0 \).

\[
\begin{align*}
\overline{a}_1 & \rightarrow \nabla \rightarrow d_3 \\
\overline{a}_0 & \rightarrow \nabla \rightarrow d_2 \\
a_1 & \rightarrow \nabla \rightarrow d_1 \\
\overline{a}_0 & \rightarrow \nabla \rightarrow d_0 \\
a_1 & \rightarrow \nabla \rightarrow d_0 \\
a_0 & \rightarrow \nabla \rightarrow d_0
\end{align*}
\]
2. Assuming that the mobility of electrons is four times as high as that of holes, design in CMOS a 4-input NAND gate with equal rise and fall times. Use symbol $w$ to indicate the width of a minimum-size nFET, use symbol $\frac{1}{2}w$ for a pFET and use symbol $\approx \frac{1}{3}w$ to designate an nFET.
Problem 13 (Core: Energy-ECE3300)  Prelim Solutions

**PROBLEM**

A 9 kVA, 208 V (L-L), 1200 rpm, 3-phase Y connected stator, synchronous generator has a stator resistance of 0.3 ohm per phase, and a synchronous reactance of 5 ohm per phase. The rotational mechanical loss is 500 W. No saturation. Field winding resistance equals 4.5 ohm. When the generator produces an output of rated kVA at rated voltage, and 0.8 pf lagging, the field current is 5 amp, determine

a) The voltage regulation

b) The efficiency
The per-phase terminal voltage: \( V_a = \frac{208}{\sqrt{3}} = 120 \, \text{V} \)

The per-phase apparent power of the generator is \( \frac{3000}{120} = 25 \, \text{kVA} \).

Hence, the rated current on a per-phase basis is \( \frac{3000}{120} = 25 \, \text{A} \).

For a lagging power factor of 0.8: \( I_a = 25 \angle -36.87^\circ \, \text{A} \).

Hence, the per-phase generated voltage is
\[
E_a = 120 \times (0.3 + j5) \times 25 \angle -36.87^\circ \\
= 229.534 \angle 25.41^\circ \, \text{V}
\]

(a) The voltage regulation is:
\[
VR\% = \frac{229.534 - 120}{120} \times 100 = 85.45\%
\]

(b) The power output: \( P_o = 3 \times 120 \times 25 \times 0.8 = 7200 \, \text{W} \)

The copper loss: \( P_{cu} = 3 \times 25^2 \times 0.3 = 562.5 \, \text{W} \)

The power developed: \( P_d = 7200 + 562.5 = 7762.5 \, \text{W} \)

The constant loss: \( P_c = 500 + 5^2 \times 4.5 = 612.5 \, \text{W} \)

Hence, the total power input: \( P_{in} = 7762.5 + 612.5 = 8375 \, \text{W} \)

The efficiency of the generator is:
\[
\eta = \frac{7200}{8375} \approx 0.86 \approx 86\%
\]
Problem 14 (Core: EDA-ECE3400)

The common source amplifier shown below is driven by a source with an internal resistance of 1 k\(\Omega\). Assuming that the amplifier is unloaded and that the DC voltage at the output is 2.5 V, compute:

- The low-frequency voltage gain of the amplifier
- The upper 3dB frequency of the voltage gain using the method of open-circuit time constants.

You may make the following simplifying assumptions:

- Neglect the effect of \(\lambda\) in the DC analysis (i.e. assume \(\lambda = 0\)).
- Neglect the body effect both in DC and in small-signal analysis.
- Assume that the gates of M1 and M2 are a small-signal ground.

Recall that for a MOSFET in saturation \(I_D = (K'/2)(W/L)(V_{GS} - V_T)^2\) and the small-signal parameters are:

\[
g_m = \sqrt{2K'(W/L)I_D} \quad r_o = 1/\left(\lambda I_D\right) \\
C_{GD} = C'_{OL} W \quad C_{GS} = C'_{OL} W + (2/3)C''_{ox} W L \\
C_{DB} = \frac{C_{db0}}{1 + |V_{DB}|/\phi_{jd}}^{m_{jd}} \quad C_{SB} = \frac{C_{sb0}}{1 + |V_{SB}|/\phi_{js}}^{m_{js}}
\]

Use the following MOSFET parameter values: **n-channel**: \(K'_n = 25 \mu A/V^2\), \(V_{TN} = 0.75\) V, \(\lambda_n = 0.01 V^{-1}\), \(C''_{ox} = 830 \mu F/m^2\), \(C_{db0} = C_{sb0} = 160 F\), \(\phi_{jd} = \phi_{js} = 0.7 V\), \(m_{jd} = m_{js} = 0.33\), \(C'_{OL} = 100 pF/m\); **p-channel**: \(K'_p = 10 \mu A/V^2\), \(V_{TP} = -0.75\) V, \(\lambda_p = 0.02 V^{-1}\), \(C''_{ox} = 830 \mu F/m^2\), \(C_{db0} = C_{sb0} = 770 F\), \(\phi_{jd} = \phi_{js} = 0.7 V\), \(m_{jd} = m_{js} = 0.33\), \(C'_{OL} = 150 pF/m\).
DC analysis
Node equation at the drain of M1:

\[
\frac{V_{D1} - 5}{25 \text{k}\Omega} + \frac{25 \cdot 10^{-6}}{2} \frac{60}{2} (V_{D1} - 0.75)^2 = 0
\]

Solve: \( V_{D1} = 1.372 \text{V} \), \( I_{D1} = 145.1 \mu\text{A} \), \( I_{D2} = I_{D3} \approx 2I_{D1} = 290.2 \mu\text{A} \). Moreover \( V_{BD3} = V_{DB2} = 2.5 \text{V} \).

Small-signal analysis

Small-signal parameters

\[
\begin{align*}
g_{m3} &= \sqrt{2K'_L(W/L)I_{D3}} = 933.1 \mu\text{A/V} \\
r_{o3} &= \frac{1}{\lambda_{p}I_{D3}} = 172.3 \text{k}\Omega \\
r_{o2} &= \frac{1}{\lambda_{n}I_{D2}} = 344.6 \text{k}\Omega \\
r_2 &= r_{o3}||r_{o2} = 114.9 \text{k}\Omega \\
A_v &= -g_{m3}r_2 = -107.2 \\
C_{GD3} &= C'_{OL}W_3 = 45 \text{fF} \\
C_{GS3} &= C'_{OL}W_3 + \frac{2}{3}C''_{oz}W_1L_1 = 377 \text{fF} \\
C_{DB3} &= \frac{C_{gso}}{(1 + V_{BD3}/\phi_{jd})^{m_jd}} = 466.3 \text{fF} \\
C_{GD2} &= C'_{OL}W_2 = 12 \text{fF} \\
C_{DB2} &= \frac{C_{gso}}{(1 + V_{DB2}/\phi_{jd})^{m_jd}} = 96.9 \text{fF} \\
C_2 &= C_{DB3} + C_{GD2} + C_{DB2} = 575.2 \text{fF}
\end{align*}
\]

The equivalent resistance across \( C_{GS3} \) is 1 k\Omega, so \( \tau_1 = 377 \cdot 10^{-15} \cdot 10^3 = 377 \text{ps} \). The equivalent resistance across \( C_2 \) is \( r_2 \), so \( \tau_2 = 575.2 \cdot 10^{-15} \cdot 114.9 \cdot 10^3 = 66.09 \text{ns} \). The equivalent resistance across \( C_{GD3} \) is computed by replacing \( C_{GD3} \) with a test current (or a test voltage).
\[
\begin{align*}
\nu_t &= \nu_{gs} + (i_t + g_m v_{gs})r_2 = r_2i_t + (1 + g_m r_2)\nu_{gs} \\
&= r_2i_t + (1 + g_m r_2)1\,k\Omega \\
R_{eq} &= \frac{\nu_t}{i_t} = r_2 + (1 + g_m r_2)1\,k\Omega = 223.1\,k\Omega \\
\tau_3 &= R_{eq}C_{GD3} = 10.04\,ns
\end{align*}
\]

Therefore the upper 3dB frequency of this amplifier is

\[
f_{3dB} = \frac{1}{2\pi(\tau_1 + \tau_2 + \tau_3)} = 2.08\,MHz
\]
i) Let \( p(\cdot) \) and \( q(\cdot) \) be Hurwitz polynomials with real coefficients of degree \( n \) with all their roots on the real axis, and no roots in common. Prove that the root locus for the open loop transfer function \( H(s) = \frac{p(s^2)}{sq(s^2)} \) cannot cross the imaginary axis, for nonzero values of the gain.

ii) Let \( 0 < z_1 < z_2 < \ldots < z_n \). Show, using root locus techniques, that the polynomial

\[
r(s; \lambda) = \lambda sq\lambda(s^2) + (1 - \lambda)p(s^2)
\]

is Hurwitz for all \( 0 < \lambda < 1 \), where

\[
p(s) = \prod_{i=1}^{n}(s + z_i), \quad q\lambda(s) = \prod_{i=2}^{n}(s + \lambda z_i + (1 - \lambda)z_{i-1}).
\]
i) Equivalently, it needs to be proven that the polynomial \( r(s; k) = sq(s^2) + kp(s^2) \) has no roots on the imaginary axis for all nonzero values of \( k \).

Since the polynomials are Hurwitz, their roots are in the open left half plane, hence on the negative real axis. This implies that \( p(s^2) \) and \( q(s^2) \) both have their roots on the imaginary axis, and by the assumption, they do not have roots in common.

Now we proceed with a proof by contradiction: Assume that \( s_0 \) is a root of \( r(s; k) \) on the imaginary axis. Thus \( s_0 = j\omega \) for some \( \omega \), and some \( k \neq 0 \). Then

\[
r(j\omega; k) = j\omega q(-\omega^2) + kp(-\omega^2) = 0,
\]

and since both \( p(-\omega^2) \) and \( q(-\omega^2) \) are real, it follows that

\[
p(-\omega^2) = \omega q(-\omega)^2 = 0.
\]

Since \( 0 \) is not a root of \( p \), \( \omega = 0 \) cannot be a solution. But this then implies that \( p(-\omega^2) = q(-\omega)^2 = 0 \), i.e., that \( p \) and \( q \) have a common root, which contradicts the assumption.

ii) Consider the polynomial

\[
R(s; \lambda, k) = p(s^2) + ksq\lambda(s^2).
\]

We will show that for \( 0 < \lambda < 1 \) and all \( k > 0 \), \( R(s; \lambda, k) \) is Hurwitz. Then surely it will be Hurwitz for \( k = \frac{\lambda}{1-\lambda} > 0 \), and since \( r(s; \lambda) = R(s; \lambda, \frac{\lambda}{1-\lambda}) \), the required result is implied.

The polynomial \( q\lambda(s) \) has roots, \(-\lambda z_i + (1 - \lambda)z_{i-1}\) on the negative real axis, and these roots are between the roots, \(-z_i\) and \(-z_{i-1}\) of \( p \). Thus \( q\lambda(s) \) and \( p(s) \) have no roots in common for all \( 0 < \lambda < 1 \).

This implies that the roots of \( sq\lambda(s^2) \) and \( p(s^2) \) are interlaced and on the imaginary axis. Consider the root locus for \( H(s) = \frac{sq\lambda(s^2)}{p(s^2)} \). As shown in (i), there are no imaginary axis crossings for \( k \neq 0 \). Consider now the pole at \( j\sqrt{z_i} \). \( n + i \) poles and zeros lie above it (on the imaginary axis) and \( n + (i - 1) \) poles and zeros beneath it. By the formula for the angles of departure, one finds

\[
\phi_\text{dep,}i = -\pi.
\]

Likewise, for every zero, \( z_i^{(\lambda)} \), of \( H(s) \), there is one more pole than zeros above and beneath it. By the formula for the angles of arrival, we get

\[
\psi_\text{arr,}i = \lim_{s \to z_i^{(\lambda)}} \arg(s - z_i^{(\lambda)}) = \pi
\]

Moreover one branch of the root locus moves towards \(-\infty \). Thus all closed loop poles, i.e. all roots of \( R(s; \lambda, k) \) are in the open left half plane for all \( 0 < \lambda < 1 \) and \( k > 0 \). Hence it can be concluded that \( r(s; \lambda) \) is Hurwitz.
Problem 16 (Specialized: Telecom-ECE3600)  Prelim Solutions

PROBLEM

When you PC needs to find out the IP address of www.cnn.com, the resolver software send a DNS lookup request (query) to a Local DNS Server. What are the three types of DNS servers (in order) that the Local DNS Server will send queries to find the answer (assuming its cache is empty)?

1. 

2. 

3. 

Identify the following parts of the URL "http://www.ece.gatech.edu/academics/index.html"

4. "gatech.edu" 

5. "www" 

6. "http" 

How does TCP know that a receiving computer has received each segment of a message stream?

7. 

What is the advantage of UDP relative to TCP for services such as DNS and NTP?

8. 

How does a CDN (like Akami) improve the delivery of Web content for its customers (like CNN)?

9. 

A "cellular" network gets its name because the area it covers is

10. 

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When you PC needs to find out the IP address of www.cnn.com, the resolver software send a DNS lookup request (query) to a Local DNS Server. What are the three types of DNS servers (in order) that the Local DNS Server will send queries to find the answer (assuming its cache is empty)?

1. _____ Root DNS Server ____________________________

2. _____ Top Level Domain Server ____________________________

3. _____ Authoritative Server ____________________________

Identify the following parts of the URL "http://www.ece.gatech.edu/academics/index.html"

4. "gatech.edu" _____ domain name ____________________________

5. "www" _____ host (server, computer) name________________________

6. "http" _____ protocol ________________________________

How does TCP know that a receiving computer has received each segment of a message stream?

7. _____ Short answer: "From the acknowledgement header." _____

    Long answer: "The ACK message from the receiver contains the byte number of the last contiguous byte received."

What is the advantage of UDP relative to TCP for services such as DNS and NTP?

8. _____ It is simple (requires less computer and network resources) _____

How does a CDN (like Akami) improve the delivery of Web content from its customers (like CNN)?

9. _____ HTTP requests are routed to the nearest CDN proxy server (which are distributed around the world). _____

A "cellular" network gets its name because the area it covers is

10. _____ Divided into cells (that can be served by low-power base stations ) ____________
Consider an air-filled parallel-plate waveguide shown below. It is possible to construct the mode electric field solution as a superposition of plane waves of equal amplitude, whose wavevectors, \( \mathbf{k}_u \) and \( \mathbf{k}_d \), are also shown. The wavevector magnitudes are identical (\( |\mathbf{k}_u| = |\mathbf{k}_d| = k = \omega/c \)).

\[ \mathbf{k}_u = k_x \mathbf{a}_y + \beta \mathbf{a}_z \quad \text{and} \quad \mathbf{k}_d = -k_x \mathbf{a}_y + \beta \mathbf{a}_z \]

The total \( y \)-directed electric field in the guide can now be written in phasor form as the sum of the two waves as:

\[ E_y(r) = E_u + E_d = E_0 e^{-jk_x r} + E_0 e^{-jk_x r} e^{j\phi} \]

where \( E_0 \) is the constant amplitude of each wave, and where the position vector is \( r = z \mathbf{a}_z + z \mathbf{a}_z \). \( \phi \) is the reflective phase shift at either boundary, which for the metal guide is equal to \( \pi \) radians.

a) Carry out the scalar products in the exponents and thus write the total field phasor in terms of \( z \) and \( z \). Simplify your result by using the Euler identity \( \sin(x) = (1/2j)(e^{jx} - e^{-jx}) \).

b) Express your above result in real instantaneous form by multiplying by \( e^{j\omega t} \) and taking the real part. Your answer should be in the form of the product of a standing wave in \( x \) and and traveling wave in \( z \).

c) Find the expression for \( k_x \) (in terms of known parameters) that will enable \( E_y \) to have the required value of zero at the guide boundaries (\( z = 0, d \)).

d) Knowing that \( k, k_x, \) and \( \beta \) form the sides of a right triangle, write an expression for \( \beta \) and from this, find an expression that will give the minimum required radian frequency \( \omega_{min} \) that will yield a propagating wave.
Consider an air-filled parallel-plate waveguide shown below. It is possible to construct the mode electric field solution as a superposition of plane waves of equal amplitude, whose wavevectors, \( \mathbf{k}_u \) and \( \mathbf{k}_d \), are also shown. The wavevector magnitudes are identical (\( |\mathbf{k}_u| = |\mathbf{k}_d| = k = \omega / c \)).

\[ \begin{array}{c}
\begin{aligned}
E_u & = E_0 e^{-jk_y y} e^{j\beta z} \\
E_d & = E_0 e^{jk_y y} e^{-j\beta z}
\end{aligned}
\end{array} \]

where \( E_0 \) is the constant amplitude of each wave, and where the position vector is \( r = x \mathbf{a}_x + z \mathbf{a}_z \). \( \phi \) is the reflective phase shift at either boundary, which for the metal guide is equal to \( \pi \) radians.

a) Carry out the scalar products in the exponents and thus write the total field phasor in terms of \( x \) and \( z \). Simplify your result by using the Euler identity \( \sin(x) = (1/2j)(e^{ix} - e^{-ix}) \).

With substitutions, we obtain

\[ E_y(x, z) = E_0 e^{-jk_y x} e^{-j\beta z} + E_0 e^{jk_y x} e^{-j\beta z} = E_0 \left[ e^{-jk_y x} - e^{jk_y x} \right] e^{-j\beta z} = -j2E_0 \sin(k_y x) e^{-j\beta z} \]

b) Express your above result in real instantaneous form by multiplying by \( e^{j\omega t} \) and taking the real part.

Your answer should be in the form of the product of a standing wave in \( x \) and and traveling wave in \( z \).

\[ \mathcal{E}(x, z, t) = \Re \left\{ E_y(x, z) e^{j\omega t} \right\} = 2E_0 \sin(k_y x) \sin(\omega t - \beta z) \]

c) Find the expression for \( k_y \) (in terms of known parameters) that will enable \( E_y \) to have the required value of zero at the guide boundaries \( (x = 0, d) \).

Need the \( \sin(k_y x) \) term to be zero at \( x = 0, d \). Therefore \( k_y = m\pi / d \), where \( m \) is an integer starting with 1.

d) Knowing that \( k, k_x \), and \( \beta \) form the sides of a right triangle, write an expression for \( \beta \) and from this, find an expression that will give the minimum required radian frequency \( \omega_{\min} \) that will yield a propagating wave.

Using the Pythagorean rule,

\[ \beta = [k^2 - k_x^2]^{1/2} = \left[ \left( \frac{\omega}{c} \right)^2 - \left( \frac{m\pi}{d} \right)^2 \right]^{1/2} \]

Requiring \( \beta \) to be real, this means that

\[ \omega_{\min} = \frac{m\pi c}{d} \]

the cutoff frequency for mode \( m \)
Problem 18 (Specialized: Optics-ECE4500)  Prelim Solutions

**Problem**

**Fabry–Pérot Interferometer for Refractive Index Measurement**

The surfaces of a slab of homogeneous material of unknown refractive index $n$ are coated with semi-reflective films to form a Fabry–Pérot interferometer. The slab has a thickness of $d = 10$ mm, and is placed in front of a convex lens with focal length $f = 300$ mm. Interference results are observed on a screen placed at the focal plane of the lens.

![Diagram of Fabry–Pérot Interferometer](image)

1. If the slab is illuminated with a plane wave of free-space wavelength $\lambda$ at normal incidence (along the optical axis of the system), find the relation between $n$ and $\lambda$ for constructive interference (bright spot) to occur on the screen.

2. If the slab is illuminated with a plane wave of free-space wavelength $\lambda$ at a small incident angle $\theta$ with respect to the optical axis of the system, find the relation between $n$, $\lambda$ and $\theta$ for constructive interference to occur on the screen.

3. Now the slab is illuminated with a sodium lamp which emits yellow light ($\lambda = 589.3$ nm) along all directions. A bright spot is formed at the screen center, and the diameter of the first bright ring is 6 mm. Determine the refractive index $n$ of the slab material.

![Diagram of Sodium Lamp and Fabry–Pérot Interferometer](image)
Fabry–Pérot Interferometer for Refractive Index Measurement

(1) The phase difference between successive transmitted beams is \( \delta = k(2d \cos \theta_t) \), where \( k = 2\pi n/\lambda \), and \( \theta_t \) is the angle of the beam within the slab. At normal incidence, \( \theta_t = \theta_i = 0 \). The construction interference condition is \( \delta = 2m\pi \), which implies
\[
d = m\frac{\lambda}{2n},
\]
where \( m \) is an integer.

(2) At oblique incidence, the constructive interference condition is \( \delta = k(2d \cos \theta_t) = 2m\pi \). From Snell’s law we have \( \sin \theta = n \sin \theta_t \). Using small angle approximation (\( \sin \theta \approx \theta \)) we have \( \theta_t \approx \theta/n \). So the constructive condition is
\[
d = m\frac{\lambda}{2n \cos(\theta/n)} \approx m\frac{\lambda}{2n(1 + \frac{\theta^2}{2n^2})}.
\]
For the last step we have used Taylor expansion \( \cos x \approx 1 - \frac{x^2}{2} \) and \( \frac{1}{1-x} \approx 1 + x \) for \( |x| \ll 1 \).

(3) Assume that the center bright spot has an order \( m \), so \( 2nd = m\lambda \).

For the first bright ring, the interference condition is \( 2nd \cos(\theta/n) = (m - 1)\lambda \).

Therefore, \( 2nd \left[ 1 - \cos \left( \frac{\theta}{n} \right) \right] = \lambda \).

Using the small angle approximation \( \left[ 1 - \cos \left( \frac{\theta}{n} \right) \right] \approx \frac{\theta^2}{2n^2} \), the equation above becomes
\[
\frac{d\theta^2}{n} = \lambda.
\]
So the refractive index is \( n = d\theta^2/\lambda \).

The radius of the ring is \( r = 3 \) mm, so the angle outside the slab (both the input and the output sides) is \( \theta = \tan^{-1} \left( \frac{r}{f} \right) \approx \frac{r}{f} = 0.01 \) rad. Finally, we find
\[
n = \frac{d\theta^2}{\lambda} = \frac{(10 \text{ mm})(0.01)^2}{(589.3 \text{ nm})} = 1.70
\]
Consider the optical data link shown below, in which the total span length is \( L = L_1 + L_2 = 150 \) km. The optical transmitter provides average power \( P_t = 10 \) dBm at carrier wavelength \( \lambda_m \); the receiver has sensitivity \( P_{rec} = -30 \) dBm, which yields a specified BER. Two single mode fibers make up the span, and these have loss and dispersion specifications as indicated. The second fiber is dispersion-compensating fiber (DCF); this fiber, while designed to have high negative dispersion, exhibits relatively high loss.

\[
\begin{align*}
10.0 \text{ dBm} & \quad D_1 = +6.0 \text{ ps/nm-km} & \quad D_2 = -100 \text{ ps/nm-km} & \quad -30.0 \text{ dBm} \\
\text{Tx} & \quad a_1 = 0.2 \text{ dB/km} & \quad a_2 = 3.0 \text{ dB/km} & \quad \text{Rx} \\
L_1 & \quad L_2
\end{align*}
\]

To achieve an overall dispersion penalty for the link that is 1 dB (the dispersion limit) or less, the path average dispersion magnitude, \(|D_{avg}|\), must be less than or equal to 2.0 ps/nm-km, where

\[
D_{avg} = \frac{D_1 L_1 + D_2 L_2}{L_1 + L_2}
\]

a. Find the lengths, \( L_1 \) and \( L_2 \), such that the link exhibits the minimum loss within the dispersion limiting constraint as defined above.

b. Determine whether or not the link is viable from a loss standpoint, as a result of your part a computations. If not, specify the minimum required gain in dB of an optical amplifier that should be positioned in line to compensate loss.

c. With the link constructed as per your results, and with the amplifier (if needed) installed, will this link be dispersion-limited, or loss-limited, or both? Explain.

d. Suppose the optical carrier wavelength is shifted to a different value, and that loss is to first order invariant with wavelength. It is known, however, that dispersion in both fibers will vary significantly with wavelength. What must be true about the relation between fiber dispersions in this link so that your previous results will hold true at all wavelengths? You should display this requirement in a simple formula.
Consider the optical data link shown below, in which the total span length is \( L = L_1 + L_2 = 150 \) km. The optical transmitter provides average power \( P_t = 10 \) dBm at carrier wavelength \( \lambda_m \); the receiver has sensitivity \( P_{rec} = -30 \) dBm. Two single mode fibers make up the span, and these have loss and dispersion specifications as indicated. The second fiber is dispersion-compensating fiber (DCF); this fiber, while designed to have high negative dispersion, exhibits relatively high loss.

\[
\begin{align*}
10.0 \text{ dBm} & \quad D_1 = +6.0 \text{ ps/nm-km} & & D_2 = -100 \text{ ps/nm-km} & & -30.0 \text{ dBm} \\
\text{Tx} & \quad a_1 = 0.2 \text{ dB/km} & & a_2 = 3.0 \text{ dB/km} & \quad \text{Rx}
\end{align*}
\]

To achieve an overall dispersion penalty for the link that is 1 dB (the dispersion limit) or less, the path average dispersion magnitude, \( |D_{avg}| \), must be less than or equal to 2.0 ps/nm-km, where

\[
D_{avg} = \frac{D_1 L_1 + D_2 L_2}{L_1 + L_2}
\]

a. Find the lengths, \( L_1 \) and \( L_2 \), such that the link exhibits the minimum loss within the dispersion limiting constraint as defined above.

We set up:

\[
|D_{avg}| L = |6.0 L_1 - 100(L - L_1)| \leq 2(150)
\]

For minimum loss, \( L_2 \) should be as short as possible, meaning that we need \( |D_{avg}| L = 300 \). Therefore, \( L_1 = 144.3 \) km, and \( L_2 = 5.7 \) km.

b. Determine whether or not the link is viable from a loss standpoint, as a result of your part a computations. If not, specify the minimum required gain in dB of an optical amplifier that should be positioned in line to compensate loss.

With the lengths as found in part a, the net loss becomes (including the 1-dB dispersion penalty):

\[
\text{Loss} = 0.2(144.3) + 3.0(5.7) + 1.0 = 47 \text{ dB}
\]

This value is greater than the allowed budget, which is 10dBm \(-(-30\text{dBm}) = 40\text{dB}\). Therefore, a 7dB amplifier is needed at minimum to satisfy the power budget.

c. With the link constructed as per your results, and with the amplifier (if needed) installed, will this link be dispersion-limited, or loss-limited, or both? Explain.

The link will be both dispersion and loss limited, because the overall length is the maximum allowable within the dispersion and loss constraints.

d. Suppose the optical carrier wavelength is shifted to a different value, and that loss is to first order invariant with wavelength. It is known, however, that dispersion in both fibers will vary significantly with wavelength. What must be true about the relation between fiber dispersions in this link so that your previous results will hold true at all wavelengths? Display this requirement in a simple formula.

The change in dispersion with wavelength in the first fiber must be just compensated by the change in dispersion in the second, leading to a path average dispersion that is invariant with wavelength. In terms of the original wavelength \( \lambda_m \) and wavelength shift \( \Delta \lambda \), we have:

\[
D_{avg} = \frac{1}{L} \left[ \left( D_1(\lambda_m) + \Delta \lambda \frac{dD_1}{d\lambda} \bigg|_{\lambda_m} \right) L_1 + \left( D_2(\lambda_m) + \Delta \lambda \frac{dD_2}{d\lambda} \bigg|_{\lambda_m} \right) L_2 \right] = 2
\]

To make this result independent of \( \Delta \lambda \), the slopes must compensate, or

\[
L_1 \frac{dD_1}{d\lambda} \bigg|_{\lambda_m} = -L_2 \frac{dD_2}{d\lambda} \bigg|_{\lambda_m}
\]
A uniform oxide layer of 0.4\(\mu\)m thickness is selectively etched to expose the bare silicon surface in some locations on a wafer (as shown below). Following the etch, a second oxidation of the wafer at 1000\(^\circ\)C in H\(_2\)O grows 0.2\(\mu\)m on the bare silicon surface region.

a) [7 pts] Following the second oxide growth, how much new silicon dioxide grows in the region where 0.4\(\mu\)m silicon dioxide thickness already exists? Assume A=0.252 \(\mu\)m and B=0.316 \(\mu\)m\(^2\)/hr.

b) [3 pts] Carefully re-sketch the wafer showing the location of the NEW oxide (due to second re-oxidation) relative to the OLD oxide (i.e., the initial 0.4\(\mu\)m oxide layer). No need to show dimensions.
**SOLUTION**

Time to grow 0.2 mm:

\[ \gamma = \frac{x^2 + A \cdot x}{B} = \frac{(0.2)^2 + (0.250)(0.2)}{0.316} = 0.386 \text{ hr} \]

Time to grow 0.4 mm:

\[ \Delta \gamma = \frac{(0.4)^2 + (0.250)(0.4)}{0.316} = 0.825 \text{ hr} \]

The extra time to grow 0.2 mm will add oxide to where we have 0.4 mm already!!

\[ \text{so we grow extra } 0.08 \text{ mm of oxide} \]

**Silicon consume:**

- 0.46 x 0.2 = 0.092 mm of Si consumed in blank area
- 0.46 x 0.08 = 0.037 mm of Si consumed in area w/ oxide

Key: New oxide & silicon interface

Original Si/SiO₂ interface
(3 pts) *Cardiovascular Anatomy.* Fill in the blanks with the *best* possible answer.

The heart has ______ chambers, named the __________ and the ___________.

The ______ side of the heart delivers blood to the lungs, and the ________ side to the rest of the body.

The heart fills during diastole, and ejects blood during ____________.

(4 pts) Draw a lumped electrical circuit model of the chamber which delivers the blood directly into the aorta, showing all of the following as circuit elements: (1) pressure filling the chamber, (2) pressure in the aorta, (3) compliance of the chamber, (4) the input valve to the chamber, and (5) the output valve from the chamber. Label your diagram completely.

(3 pts) Stroke volume is the volume of blood pumped by the heart in a single heartbeat, and is an important parameter to measure clinically. From one heartbeat to the next, stroke volume is not constant, and actually varies due to the dynamics of breathing.

Consider first the impact of breathing on the filling of the heart: do you expect the heart to receive more blood back from the veins during peak inspiration or peak expiration? Why? (Consider the pressures in the chest cavity, and how they change with respiration.)
SOLUTION

(3 pts) Cardiovascular Anatomy. Fill in the blanks with the best possible answer.

The heart has ___four___ chambers, named the ___atria___ and the ___ventricles___.

The ___right___ side of the heart delivers blood to the lungs, and the ___left___ side to the rest of the body.

The heart fills during diastole, and ejects blood during ___systole___.

(4 pts) Draw a lumped electrical circuit model of the chamber which delivers the blood directly into the aorta, showing all of the following as circuit elements: (1) pressure filling the chamber, (2) pressure in the aorta, (3) compliance of the chamber, (4) the input valve to the chamber, and (5) the output valve from the chamber. Label your diagram completely.

(3 pts) Stroke volume is the volume of blood pumped by the heart in a single heartbeat, and is an important parameter to measure clinically. From one heartbeat to the next, stroke volume is not constant, and actually varies due to the dynamics of breathing.

Consider only the impact of breathing on the filling of the heart: do you expect the heart to receive more blood back from the veins during peak inspiration or peak expiration? Why? (Consider the pressures in the chest cavity, and how they change with respiration.)

During peak inspiration, the chest cavity drops to a lower pressure compared to atmospheric pressure, to such air into the lungs; during peak expiration, the diaphragm contracts and increases the pressure in the chest cavity compared to atmospheric. Thus we would expect increased filling of the heart during peak inspiration, where the pressure difference between the central venous system and the right ventricle is maximal.
1) The organism in question is a (hypothetical) squid living below the ice cap of Europa (also hypothetical that there would be water in liquid form there). He is an interplanetary friend of SpongeBob. He has a body temperature 3°C above the freezing point of water.

   a) Calculate the Nernst potential for each of the following ions:

   - \([K^+]_i = 150 \text{ mM} \quad [K^+]_e = 10 \text{ mM}\)
   - \([Na^+]_i = 12 \text{ mM} \quad [Na^+]_e = 150 \text{ mM}\)
   - \([Cl^-]_i = 25 \text{ mM} \quad [Cl^-]_e = 170 \text{ mM}\)
   - \([Ca^{2+}]_i = 180 \text{ mM} \quad [Ca^{2+}]_e = 20 \text{ mM}\)

**Constants:**

- \(R = 8.314 \frac{\text{J}}{\text{K}\text{mol}}\)
- \(F = 96485 \frac{\text{C}}{\text{mol}}\)
- Avogadro’s Number = \(6.022 \times 10^{23} \frac{\text{atoms}}{\text{mol}}\)
Through Hodgkin-Huxley type experiments it is determined that the membrane conductance for Cl$^-$ and Ca$^{2+}$ vary with time and voltage during the action potential. The conductances for K$^+$ and Na$^+$, however, are non-zero and do not change with time or voltage.

b) Sketch the full circuit model for the membrane, including all components. Explain the physiological origin of each of the components. What kind of protein channels must be present for each ion identified?
c) Write the differential equation for the total membrane current associated with the model.

d) For this organism, you find that the early part of the action potential is due to an influx of Cl$^-$ and the latter part of the action potential is due to an efflux of Ca$^{2+}$. Speculate on the differences you would expect to find in the amino acid residue types in the voltage gated Ca$^{2+}$ channels and the voltage-gated Cl$^-$ channels.
2) Using the following resting channel conductance and density values for a mammalian axon, calculate the sodium and potassium membrane conductance of a section which has a diameter of 0.1 mm and a length of 1 mm.

- \( \text{Na}^+ \): channel conductance of 10 pS; channel density of 1000 per \( \mu \text{m} \)
- \( \text{K}^+ \): channel conductance of 5 pS; channel density of 300 per \( \mu \text{m} \)
3) Compare and contrast semiconductor pn junctions and electrically active membranes. Be thorough in your comparison, indicating what is remarkably similar and what is not. Also include a description of the physiological make-up of the plasma membrane, i.e. draw a picture.
4) Equation (1) is for Johnson noise. At a membrane voltage, $V_m$, of 70 mV and a Nernst potential of 20 mV for Na$^+$, what will be the membrane current through a single Na$^+$ channel. Also calculate the signal to noise ratio of that channel in dB.

$$i_n = \sqrt{\frac{4K_B T \Delta f}{R}}$$  \hspace{1cm} (1)

**Constants**

- $K_B = 1.380 \times 10^{23} \frac{J}{K}$
Problem 22 (Specialized: BioEng-ECE4784)  Prelim Solutions

**SOLUTION**

\[ T = 3^\circ C + 273 = 276 \text{ K} \]
\[ \frac{RT}{F} = \frac{8.314 \text{ J}}{276 \text{ K}} \]
\[ 96.485 \text{ C/mol} \]
\[ \begin{align*}
V_K &= 2.38 \ln \left( \frac{10 \text{ mM}^2}{150 \text{ mM}} \right) = -6.45 \text{ mV} \\
V_{Na} &= 2.38 \ln \left( \frac{150 \text{ mM}}{1 \text{ L}} \right) = +60 \text{ mV} \\
V_{Cl} &= -2.38 \ln \left( \frac{170 \text{ mM}}{25 \text{ mM}} \right) = -45.6 \text{ mV} \\
V_{Ca} &= 2.38 \ln \left( \frac{20 \text{ mM}}{100 \text{ mM}} \right) = -26.1 \text{ mV}
\end{align*} \]

(b) Sketch of the ion channel:

- \( g_K, g_{Na}, g_{Cl} \) represent the resting channels for \( Na^+ \) and \( K^+ \)
- \( g_{Ca}, g_a \) represent voltage-gated and resting channels for \( Cl^- \) and \( Ca^{2+} \)
- \( \frac{1}{T} \) are the Nernst potentials calculated in (a)
- \( \frac{1}{T} \) denotes an activation coefficient of the phospholipid bilayer
- Ion pump... pumps \( Ca^{2+} + Cl^- \)
- \( Cl^- \text{ out} + \text{Ca}^{2+} \text{ in} \)
d) \[ I_{\text{total}} = g_k (V_m - V_e) + g_{Na} (V_m - V_n) + g_{Cl} (V_m - V_{Cl}) + g_{Ca} (V_m - V_{Ca}) + C_m \frac{dV_m}{dt} + \ell p \]

C) voltage gate channels for Cl-
- faster, so likely to have
  - greater dipole moment
  - charge
  - lower mass/momentum of inertia

Voltage rate channels for Ca^2+
- slower - so likely to have
  - less charge
  - lower dipole moment
  - greater mass/momentum of inertia