INSTRUCTIONS

1. Please check to ensure that you have a complete exam booklet. There are 22 numbered problems. Note that Problem 2 occupies 2 pages, Problem 3 occupies 2 pages, Problem 12 occupies 2 pages, Problem 20 occupies 2 pages, Problem 22 occupies 3 pages. Including the cover sheet, you should have 58 pages. There should be no blank pages in the booklet.

2. The examination is closed book and closed notes. No reference material is allowed at your desk. A calculator is permitted.

3. All wireless devices must be turned off for the entire duration of the exam.

4. You may work a problem directly on the problem statement (if there is room) or on blank sheets of paper available from the exam proctor. Do not write on the back side of any sheet.

5. Your examination code number MUST APPEAR ON EVERY SHEET. This includes this cover sheet, the problem statement sheets, and any additional work sheets you turn in. DO NOT write your name on any of these sheets. Use the preprinted numbers whenever possible, or WRITE LEGIBLY!!

6. Under the rules of the examination, you must choose 8 problems to be handed in for grading. Each problem to be graded should be separated from the rest of the materials, stapled to the associated worksheets, and placed on the top of the appropriate envelope in the front of the exam room. DO NOT TURN IN ANY SHEETS FOR THE OTHER 14 PROBLEMS!!

7. The examination lasts 4 hours, from 9:30 AM to 1:30 PM EST.

8. When you hand in the exam:

   (a) Separate the 8 problems to be graded as explained above.
   (b) Check to see that your Code Number is in EVERY sheet you are turning in.
   (c) On the section at the bottom of this page, CIRCLE the problem numbers that you are turning in for grading.
   (d) Turn in this cover sheet (containing your code number) and the 8 problems to be graded.
   (e) All other material is to be placed in the discard box at the front of the room. You are not allowed to take any of the exam booklet pages from the room!
Problem 1 (Core: VLSI - ECE 2020)  Solution

PROBLEM

Answer all the parts of the question.

a. (2pts each) Use algebraic techniques to simplify:
   
i. \( F = X(A + B + CDE + F\bar{G} + W + X + YZ) \)
   
ii. \( F = AB + \bar{A}C + BCD \)

b. (3pts) Consider the following Boolean Function:
   \[ F_{(A,B,C,D,E)} = \overline{(A + B)(C + D) + (C + D)} + E \]
   Use mixed logic design style to implement the function using OR gates and INVERTERs only

c. (3pts) Consider a latch with an input signal (In) and a enable signal (Enable). The latch is transparent when the enable signal (Enable) is high and is blocking when the enable signal (Enable) is low. Draw the timing diagram corresponding to the input pattern (In) and the Enable signal (Enable) as shown below. Consider zero delay between the input and the output.

![Timing Diagram](image-url)
Solution

a.  

i. \( F = X(A + B + CDE + F\bar{G} + W + X + YZ) \)
\[ = XA + XB + XCDE + XFG + XW + XX + XYZ \]
\[ = XA + XB + XCDE + XFG + XW + X + XYZ \]
\[ = X(A + B + CDE + F\bar{G} + W + 1 + YZ) \]
\[ = X \]

ii. \( F = AB + \bar{A}C + BCD \)
\[ = AB + \bar{A}C + BC + BCD \quad \text{[Consensus Theorem]} \]
\[ = AB + \bar{A}C + BC(1 + CD) \]
\[ = AB + \bar{A}C + BC(1) \]
\[ = AB + \bar{A}C + BC \]
\[ = A.B + \bar{A}.C \quad \text{[Consensus Theorem]} \]

b. 

![Logic Diagram](image1)

c. 

![Waveform Diagram](image2)
Problem 2 (Core: DSP - ECE 2026) Solution

**Problem**

*When answering the five parts on the next page, consult the following resource:*

Each of the **magnitude only** frequency response plots (A–F) in the figure below corresponds to one of the systems $S_1 - S_7$, specified by either a system function $H(z)$ or a difference equation.

![Frequency response plots](image)

*Note: The frequency axis for each plot extends over the range $-\pi \leq \omega \leq \pi$.  

$S_1$ : $y[n] = 0.6y[n - 1] + 4x[n] - 4x[n - 1]$

$S_2$ : $y[n] = 0.8y[n - 1] + 0.5x[n] + 0.5x[n - 1]$

$S_3$ : $H(z) = \frac{0.5(1 - z^{-1})}{1 + 0.8z^{-1}}$

$S_4$ : $y[n] = \frac{1}{2}x[n] + \frac{5}{4}x[n - 1] + \frac{3}{2}x[n - 2] + \frac{5}{4}x[n - 3] + \frac{1}{2}x[n - 4]$

$S_5$ : $H(z) = 1 - z^{-1} + z^{-2} - z^{-3} + z^{-4}$

$S_6$ : $y[n] = \sum_{k=0}^{3} x[n - k]$

(a) For Frequency Response (A), determine the output when the input is $x[n] = 3$ for $-\infty < n < \infty$.

(b) For Frequency Response (B), determine the impulse response of the system.

(c) Determine which system has the frequency response (C).

(d) For Frequency Response (E), make a pole-zero plot for the corresponding system.

(e) The frequency response (magnitude) of system $S_4$ is the plot shown in (F). The impulse response of $S_4$ is $h_4[n]$ and the length of $h_4[n]$ is finite. If $h_4[n]$ is zero-padded to a length of 100, and a 100-point DFT is computed, the result is $H_{100}[k]$. Explain how to use the plot in (F) to obtain an estimate of the magnitude of the DFT at $k = 90$ without computing the DFT.

$$|H_{100}[90]| \approx 4$$
(a) For Frequency Response (A), determine the output when the input is $x[n] = 3$ for $-\infty < n < \infty$.

This is system $S_2$, so $\omega H_1(e^{j\omega}) = 0$ and the DC value of the frequency response is $H_1(e^{j\omega}) = 5$. Thus, $y[n] = (5)(3) = 15$ for $-\infty < n < \infty$.

(b) For Frequency Response (B), determine the impulse response of the system.

This is $S_3$ and the system function $H_3(z)$ is given, so take the inverse $z$-transform:

$$H_3(z) = \frac{0.5(z - 1)}{1 + 0.8z^{-1}} = \frac{0.5}{1 + 0.8z^{-1}} - \frac{0.5z^{-1}}{1 + 0.8z^{-1}} = 0.5 - 0.9z^{-1}$$

$$\Rightarrow \quad h_3[n] = 0.5(-0.8)^n u[n] - 0.5(-0.8)^{n-1} u[n-1]$$

or, $h_3[n] = 0.5\delta[n] - 0.9(-0.8)^{n-1} u[n-1]$

(c) Determine which system has the frequency response (C).

The answer is $S_7$. The frequency response shape is a Dirichlet form which is characteristic of a running-sum filter. To verify take the $z$-transform

$$H_7(z) = \sum_{k=0}^{4} z^{-k} = \frac{1 - z^{-5}}{1 - z^{-1}}$$

$$\Rightarrow \quad \text{zeros at } z = e^{j2\pi/5}, \ell = 1, 2, 3, 4. \text{ No zero at } z = 1 \text{ which is } \omega = 0.$$  

(d) For Frequency Response (E), make a pole-zero plot for the corresponding system.

This is system $S_4$, Four zeros on the unit circle at $e^{j0.2\pi}$, $e^{j0.6\pi}$ and four poles at $z = 0$.

(e) The frequency response (magnitude) of system $S_4$ is the plot shown in (F). The impulse response of $S_4$ is $h_4[n]$ and the length of $h_4[n]$ is finite. If $h_4[n]$ is zero-padded to a length of 100, and a 100-point DFT is computed, the result is $H_{100}[k]$. Explain how to use the plot in (F) to obtain an estimate of the magnitude of the DFT at $k = 90$ without computing the DFT.

$$|H_{100}[90]| \approx 3.8 \pm 0.1$$

The DFT samples the DTFT at $\omega = 2\pi k/100$ for $k = 0, 1, \ldots, 99$, so we want the value of the DTFT at $\omega = 2\pi(90)/100 = 1.8\pi$. On the frequency response plot (F) frequencies wrap around modulo-2$\pi$, i.e., $\omega = 1.8\pi - 2\pi = -0.2\pi$, so the magnitude $|H_{100}[90]|$ is approximately equal to 3.8 or 3.9, i.e., a little bit less than 4.
Problem 3 (Core: CSS - ECE 2036) Solution

What is printed by the program below?

```c++
#include <iostream>
using namespace std;
class Base {
    // Define a base class
    public:
    virtual void Sub1() = 0;
    void Sub2();
    virtual void Sub3();
    virtual void Sub4();
};
class A : public Base {
    // Class A derives from Base
    public:
    void Sub1();
    void Sub2();
    void Sub4();
};
class B : public Base {
    // Class B derives from Base
    public:
    void Sub1();
    void Sub2();
    void Sub3();
};

// Base Class Methods
void Base::Sub2() { cout << "Hello from Base::Sub2()" << endl; }
void Base::Sub3() {
    cout << "Hello from Base::Sub3()" << endl;
    Sub1(); // DON'T MISS THIS CALL IN YOUR ANSWER
    Sub4(); // DON'T MISS THIS CALL IN YOUR ANSWER
}
void Base::Sub4() { cout << "Hello from Base::Sub4()" << endl; }

// Class A Methods
void A::Sub1() { cout << "Hello from A::Sub1()" << endl; }
void A::Sub2() { cout << "Hello from A::Sub2()" << endl; }
void A::Sub4() { cout << "Hello from A::Sub4()" << endl; }

// Class B Methods
void B::Sub1() { cout << "Hello from B::Sub1()" << endl; }
void B::Sub2() { cout << "Hello from B::Sub2()" << endl; }
void B::Sub3() { cout << "Hello from B::Sub3()" << endl; }

// A Helper Subroutine
void SubP(Base* x) {
    x->Sub1();
    x->Sub3();
    x->Sub4();
} // CONTINUED ON NEXT PAGE
```

Program q5.cc
Problem 3 (Core: CSS - ECE 2036) Solution

57 // Another helper
58 void SubR(Base& x)
59 {
60   x.Sub2();
61   x.Sub3();
62   x.Sub4();
63 }
64 // Another Helper
65 void SubV(B b0)
66 {
67   b0.Sub1();
68   b0.Sub2();
69 }
70
71 int main()
72 {
73   A a;
74   B b;
75   SubP(&a);
76   SubR(b);
77   SubV(b);
78 }

Program q5.cc (continued)
Solution:
Hello from A::Sub1()
Hello from Base::Sub3()
Hello from A::Sub1()
Hello from A::Sub4()
Hello from A::Sub4()
Hello from Base::Sub2()
Hello from B::Sub3()
Hello from Base::Sub4()
Hello from B::Sub1()
Hello from B::Sub2()
The circuit shown below is at steady state before the switch closes. Determine the capacitor voltage $v(t)$ for $t > 0$. 

![Circuit Diagram]
1. Find the initial conditions:

Use NVA:

\[
\frac{20 - V_a}{10} \times 4 = \frac{V_a}{10}
\]

\[V_a = 16 \text{ V}\]

The initial conditions are

\[i(0^-) = \frac{V_a}{10} = 1.6 \text{ A}\]

\[v(0^-) = 0 \text{ V}\]

2. After the switch closes, find the Thévenin equivalent circuit for the part of the circuit to the left of nodes a and b.

\[i_a + 3i_a = 0 \implies i_a = 0\]

\[\implies V_{oc} = 20 \text{ V}\]

\[i_a = \frac{20 \text{ V}}{10 \Omega} = 2 \text{ A}\]

\[i_{sc} = i_a + 3i_a = 8 \text{ A}\]

\[R_t = \frac{V_{oc}}{i_{sc}} = \frac{20 \text{ V}}{8 \text{ A}} = 2.5 \Omega\]
Use Kirchhoff's Current Law (KCL) to get
\[ i(t) = \frac{v(t)}{R} + C \frac{d}{dt} v(t) \]

Use Kirchhoff's Voltage Law (KVL) to get
\[ v_{oc} = R_t i(t) + L \frac{d}{dt} i(t) + v(t) \]

Substitute to get
\[
\begin{align*}
    v_{oc} &= \frac{R_t}{R} v(t) + R_t C \frac{d}{dt} v(t) + \frac{L}{R} \frac{d}{dt} v(t) + LC \frac{d^2}{dt^2} v(t) - v(t) \\
    &= LC \frac{d^2}{dt^2} v(t) + \left( R_t + \frac{L}{R} \right) \frac{d}{dt} v(t) + \left( \frac{R_t}{R} + 1 \right) v(t)
\end{align*}
\]

i.e.,
\[
\frac{d^2}{dt^2} v(t) + \left( \frac{R_t}{L} + \frac{1}{RC} \right) \frac{d}{dt} v(t) + \frac{R_t + R}{RLC} v(t) = \frac{v_{oc}}{LC}
\]

Compare to
\[
\frac{d^2}{dt^2} v(t) + 2\alpha \frac{d}{dt} v(t) + \omega_0^2 v(t) = f(t)
\]

to get
\[
\begin{align*}
    2\alpha &= \frac{R_t}{L} + \frac{1}{RC} \\
    \omega_0^2 &= \frac{R_t + R}{RLC} \quad \text{and} \quad f(t) = \frac{v_{oc}}{LC} \\
    \alpha &= 5.125 \\
    \omega_0 &= 10.25 \\
    \frac{v_{oc}}{LC} &= 2000
\end{align*}
\]
The roots of the characteristic equation are

\[ s_\pm = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \]

\[ = -5.125 \pm j 9.9365 \]

The natural response is

\[ v_n(t) = e^{-5.125t} \left( B_1 \cos(9.9365t) + B_2 \sin(9.9365t) \right) \]

The forced response is

\[ v_f(t) = 16 \quad V \]

The complete response is

\[ v_c(t) = 16 + e^{-5.125t} \left( B_1 \cos(9.9365t) + B_2 \sin(9.9365t) \right) \]

Evaluate \( B_1 \) and \( B_2 \) using the initial conditions.

\[ v(0) = 16 + B_1 = 0 \quad \Rightarrow \quad B_1 = -16 \]

\[ i(0) = C \frac{d}{dt} v(0) = -5.125 B_1 + 9.9365 B_2 = 1.6 \]

\[ \Rightarrow B_2 = -8.0914 \]

Finally

\[ v(t) = 16 - e^{-5.125t} \left( 16 \cos(9.9365t) + 8.0914 \sin(9.9365t) \right), \quad t > 0 \]
Consider the problem of turning an ordinary array of numbers into a heap.

a) Describe in words an efficient method to perform this conversion. Your method should be faster than simply inserting all elements from the array into a heap. Analyze the running time of your solution for an $n$-element heap and compare it to the running time of the method that inserts all of the elements one-by-one into a heap.

b) Execute your method on the below array. Show several intermediate stages of your method on the array (using any representation you choose) but draw the final heap using the array implementation of a heap.

```
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
5 13 17 2 9 8 26 19 31 8 4 16 _ _ _
```
a) "Heapify" the array by bubbling down the 1st half of its elements. To bubble down an element, if the element is less than either of its children, swap it with its larger child and repeat this operation until the element is greater than or equal to its children or it becomes a leaf.

\[
\text{no. of operations} = \frac{n}{4} \cdot 1 + \frac{n}{8} \cdot 2 + \ldots + 1, \log_2 n
\]

No. of operations is \(O(n)\) while inserting every element in a heap is \(O(n \log n)\).

b) Initial heap

\[\begin{array}{cccc}
5 & 13 & 17 & \\
19 & 9 & 8 & 26 \\
31 & 4 & 16 & \\
\end{array}\]

After level 1 nodes are bubbled down:

\[\begin{array}{cccc}
5 & 13 & 17 & \\
19 & 2 & 8 & 48 \\
31 & 16 & 4 & \\
\end{array}\]

After level 1 nodes are bubbled down:

\[\begin{array}{cccc}
5 & 13 & 17 & \\
19 & 2 & 8 & 48 \\
31 & 16 & 4 & \\
\end{array}\]

After root is bubbled down (final):

\[\begin{array}{cccc}
31 & 19 & 26 & 13 \\
9 & 16 & 17 & 5 \\
2 & 8 & 4 & 8 \\
\end{array}\]

Final heap as an array:
Consider the below circuit. A long transmission line with impedance 50 Ω carries an AC steady state sinusoidal voltage from a high power generator to a RADAR antenna. The antenna is represented as a 200 Ω resistor. To eliminate reflections that could return to the generator and damage it, a stub is inserted in between the line and load that is exactly one quarter of a wavelength long, with impedance $Z_0$. This technique is called quarter-wave matching.

(a) What value of $Z_0$ ensures perfect matching, i.e., none of the power from the generator returns back to it?

(b) What is the voltage standing wave ratio (VSWR) in the 50 Ω line, and in the quarter-wave stub?

(c) Using the principles of transmission line theory, explain qualitatively why this technique is successful in eliminating reflections. How does it work?
(a) To eliminate reflections, we need the input impedance at the junction looking to the right to be 50 Ω, so that the reflection coefficient is 0. The formula for the input impedance of a quarter-wavelength stub is $Z_{in} = \frac{Z_0^2}{Z_L}$. In this case, $Z_L = 200 \, \Omega$ and $Z_{in} = 50 \, \Omega$. Solving this equation gives $Z_0 = 100 \, \Omega$

(b) The voltage standing wave ratio in the 50 Ω line is 1 because there are no reflections and thus no standing wave. The voltage standing wave ratio in the 100 Ω stub is calculated from $VSWR = \frac{1 + |\Gamma|}{1 - |\Gamma|}$ where $\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = -\frac{1}{3}$, so $VSWR = 2$ in the stub.

(c) There are a few possible ways to state this. When voltage first reaches the load, some is reflected and some is transmitted into the stub. The voltage that goes into the stub then propagates to the load, reflects, and returns back to the junction. The back and forth reflections all reach the junction, and some voltage is then transmitted across back into the main line. The amplitude of this returning transmission is the same as what originally reflected from the junction, but the phase is 180° off, so they cancel out and there is no net reflection.

Another possible explanation is to look at the reflection coefficient at each end of the stub. They are of the same magnitude but opposite in sign. As such, the stub is equally unmatched to both the load and the input line, but in opposite senses.

You can also explain the fact that a quarter wave segment means that the phase shift for a back-and-forth propagation is 180°, in addition to the phase shift generated by reflection at the load. But the reflection coefficient for current is −1 times the reflection coefficient for voltage. So at quarter wavelength, a short circuit becomes an open circuit and vice versa. For this case, a load that is double the impedance of the stub has an input impedance that is half the impedance of the stub. So the stub line is at the geometric mean between the load and the line.
You are given a system with a clock rate of 1 GHz. Registers have a setup and hold time of 5 ps. The settling time constant for the registers is $\tau = 10 \text{ ps}$. What is the probability of failure if the signal is given one clock period to resolve?
You are given a system with a clock rate of 1 GHz. Registers have a setup and hold time of 5 ps. The settling time constant for the registers is $\tau = 10 \text{ ps}$. What is the probability of failure if the signal is given one clock period to resolve?

$$P_F = P_E P_S$$

$$= \frac{t_{SH}}{T} e^{-S/\tau}$$

$$= \frac{0.05\text{ns}}{1\text{ns}} e^{-1\text{ns}/10\text{ps}} = 1.86 \times 10^{-45}$$
A non-ideal operational amplifier shown here has an open loop gain at low frequencies of 100 \( \text{v/v} \), an input impedance of 20K ohms, an output impedance of 50 ohms and an open loop low pass bandwidth of 200 Hz. This amplifier is used in a non-inverting configuration to achieve a low frequency voltage gain of 10 \( \text{v/v} \). If \( R_2 = 91 \text{K ohms} \):

a) (4 points) Showing your work for full credit, what is the required value of \( R_1 \)?

b) (2 points) Showing your work for full credit, what is the new closed loop amplifier bandwidth?

c) (2 points) Showing your work for full credit, what is the new closed loop amplifier input impedance?

d) (2 points) Showing your work for full credit, what is the new closed loop amplifier output impedance?
Background Material if feedback equation is not known.

\[
v_+ = v_{in} - \frac{R_1}{R_1 + R_2} = \beta v_{out}
\]

where \( \beta = \frac{R_1}{R_1 + R_2} \) is known as the feedback factor

\[
v_{out} = A_{\text{openloop}}(v_+ - v_-) = A_{\text{openloop}}(v_+ - \beta v_{out})
\]

so since \( v_- = v_{in} \),

\[
A_{v, \text{closedloop}} = \frac{v_{out}}{v_{in}} = \frac{A_{\text{openloop}}}{1 + \beta A_{\text{openloop}}}, \text{where } \beta A_{\text{openloop}} \text{ is the loop gain}
\]

\[
10 = \frac{100}{1 + \beta} \quad \Rightarrow \beta = \frac{9}{100} = \frac{R_1}{R_1 + R_2}
\]

\[
\frac{9}{100} R_1 + \frac{9}{100} R_2 = R_1
\]

\[
R_1 = \frac{9}{91} R_2 = 9K
\]

\[
f_{\text{closedloop}} = (1 + \beta A_{\text{openloop}}) f_{\text{openloop}} = (1 + 0.09 \times 100) \times 200 = 2000 \text{Hz}
\]

\[
R_{in, \text{closedloop}} = (1 + \beta A_{\text{openloop}}) R_{in, \text{openloop}} = (1 + 0.09 \times 100) \times 20K = 200K \Omega
\]

\[
R_{out, \text{closedloop}} = \frac{R_{out, \text{openloop}}}{(1 + \beta A_{\text{openloop}})} = \frac{50}{(1 + 0.09 \times 100)} = 5 \Omega
\]
Alternative solution for parts (a-d) accounting for all non-ideal components. Credit given for short (previous) and long answers.

1) \[ \frac{V_{in} - V_i}{20k} + \frac{V_{out} - V_i}{91k} - \frac{V_i}{R_1} = 0 \]
\[ i_1 + i_3 = 0 \]

2) \[ V_{out} = 100(V_{in} - 0)(\frac{V_{out} - V_i}{91k})50 \] eliminate \( V_i \)

\[ \frac{V_{out}}{i_3} = (1 - \frac{50}{91k}) - 100V_{in} = -100V_{in} - \frac{50V_{in}}{91k} \]

4) \[ V_i = \frac{V_{out} (1 - \frac{50}{91k}) - 100V_{in}}{[100 + \frac{50}{91k}]} \]

4) + 3) \[ V_i = \frac{V_{in} \left[ 10\left(1 - \frac{50}{91k}\right) - 100\right]}{[100 + \frac{50}{91k}]} \]

5) \[ V_i = 0.9V_{in} \]

5) + 1) + 3) \[ V_{in} \left(1 - \frac{0.9}{20k}\right) + \frac{V_{out}}{91k} - \frac{V_{in}(0.9) - V_{in}(0.1)}{R_1} = 0 \]

\[ V_{out} = 91k \left[ \frac{0.9}{91k} + \frac{0.9}{R_1} - \frac{0.1}{20k} \right] V_{in} \]

\[ \frac{V_{out}}{V_{in}} = 10 = 91k \left[ \frac{0.9}{91k} + \frac{0.9}{R_1} - \frac{0.1}{20k} \right] \]

\[ \frac{1}{R_1} = 0.000116 \]

\[ R_1 = 857.1 \Omega \]
b) Same as above

c) \[ R_{in} = \frac{V_{in}}{I_{1}} = \frac{V_{in}}{(V_{in} - V_{1})/20k} \] = \[ \frac{20k}{0.1} = 200 \text{ k}\Omega \]

d) \[ R_{out} = \frac{V_{out}}{I_{out}} \left|_{V_{in}: 0} \right. \]

1) \[ V_{out} = V_{1} + I_{out1} (91k) \]
   \[ = I_{out1} (20k || 8.571k + 91k) \Rightarrow R_{out1} = (20k || 8.571k + 91k) \]

2) \[ -100V_{1} + I_{out2} (50) = V_{out} \]

3) \[ V_{1} = V_{out} \left( \frac{20k || 8.571k}{20k || 8.571k + 91k} \right) \]

2) + 3) \[ \Rightarrow I_{out2} = \frac{V_{out}}{50} \left[ 1 + \frac{20k}{20k || 8.571k + 91k} \right] \]

\[ R_{out} = \frac{V_{out}}{I_{out1} + I_{out2}} = \frac{V_{out} \left( \frac{1}{20k || 8.571k + 91k} \right) + V_{out} \left[ 1 + \frac{100(20k || 91k)}{20k || 8.571k + 91k} \right]}{50} \]

\[ R_{out} = 6.95 \text{ k}\Omega \]
Consider the 3-phase, 230 V, 50 kVA, load operating at power factor 0.8 (lagging), which as shown in the following figure, is supplied by three 20 kVA, 1330/230 V, 50 Hz transformers connected in Y-Δ by means of a common 3-phase feeder whose impedance is 0.002 + j0.01 Ohms per phase. The transformers are supplied from a 3-phase source through a 3-phase feeder whose impedance is 0.5 + j5 Ohms per phase. The equivalent impedance of one transformer referred to the low-voltage side is 0.1 + j0.25 Ohms. Determine the required supply voltage (phase-to-phase) if the load voltage is 230 V.
Problem 9 (Core: POWER - ECE 3072)

Solution

**Problem XX (Core: Energy-ECE3072) Code Number: ______

1

SOLUTIONS**

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**Ed Impedance of the transformer referred to HV side:**

\[
\text{Reg HV} + j \times \text{Eq HV} = \left( \frac{1330}{230} \right)^2 (0.10 + j0.25) = 3.34 + j8.36
\]

**Turn ratio of the equivalent Y-Y bank:**

\[
a' = \sqrt{3} \times \frac{1330}{230} = 10
\]

**Single-phase equivalent circuit shown in pic. By referring to LV side (second pic), we have:**

\[
\begin{align*}
R &= (0.5 + 3.34) \frac{1}{\sqrt{2}} + 0.002 = 0.0404 \text{ S} \\
X &= (5 + 8.36) \frac{1}{\sqrt{2}} + 0.01 = 0.1436 \text{ H} \\
V_L &= \frac{230}{\sqrt{3}} \times 0^\circ = 133 \times 0^\circ \text{ V} \\
I_L &= \frac{5}{3 \times 133} = 125.5 \times \frac{1}{\text{KVA}}
\end{align*}
\]

\[
\phi_L = -\cos^{-1}(0.8) = -36.9^\circ
\]

\[
V_s' = 133 \times 0^\circ + (125.5 \times -36.9^\circ)(0.0404 + j0.1436) \\
= 148.3 \times 4.4 \text{ V}
\]

\[
\Rightarrow V_s' = \sqrt{3} a' V_s' = 2568.6 \text{ V}
\]
Problem 10 (Core: DSP/TLCOM - ECE 3077) Solution

Problem

Light bulbs manufactured by a certain company are known to last 200 hours on the average. It is assumed that the lifetime $T$ of a particular light bulb has an exponential distribution, i.e., its pdf is of the form

$$f_T(t) = \alpha e^{-\alpha t}, \quad t \geq 0.$$ 

a) What fraction of the bulbs manufactured will last longer than 250 hours?

b) Bulbs are tested one after the other to see if they last longer than 250 hours. Let $X$ be the number of trials to obtain the tenth bulb lasting longer than 250 hours. What is the probability that $X = 20$?

c) Bulbs cost $1.25 to manufacture and are sold for $2.50. If a bulb burns out before 100 hours the customer is entitled to a full refund. What is the expected profit per bulb?
Light bulbs manufactured by a certain company are known to last 200 hours on the average. It is assumed that the lifetime $T$ of a particular light bulb has an exponential distribution, i.e., its pdf is of the form

$$f_T(t) = \alpha e^{-\alpha t}, \quad t \geq 0.$$  

a) What fraction of the bulbs manufactured will last longer than 250 hours?

Answer

$$P[T > 250] = \int_{250}^{\infty} \alpha e^{-\alpha t} dt$$

$$= -e^{-\alpha t} \bigg|_{250}^{\infty}$$

$$= e^{-250\alpha}$$

However,

$$E[T] = \int_{0}^{\infty} t \alpha e^{-\alpha t} dt$$

$$= \frac{1}{\alpha} = 200$$

Hence, $\alpha = 1/200$ and

$$P[T > 250] = e^{-250/200} = e^{-5/4} = 0.2865$$

b) Bulbs are tested one after the other to see if they last longer than 250 hours. Let $X$ be the number of trials to obtain the tenth bulb lasting longer than 250 hours. What is the probability that $X = 20$?

Answer

From part a) $P[T > 250] = e^{-5/4}$ for each trial.

Consider a point in the sample space that achieves the tenth success on the twentieth trial. The probability of any such point in the sample space is

$$(e^{-5/4})^{10} (1 - e^{-5/4})^{10}$$

since it contains 10 successes and 10 failures.
Since one of the 10 successes occurs on the twentieth trial, there are 9 successes distributed in some manner among the first 19 trials. There are \( \binom{19}{9} \) ways to distribute the 9 successes in some manner among the first 19 trials. Therefore,

\[
P[X = 20] = \binom{19}{9}(e^{-5/4})^{10}(1 - e^{-5/4})^{10} \]

\[
= 92378(e^{-5/4})^{10}(1 - e^{-5/4})^{10} \]

\[
= .01177
\]

c) Bulbs cost $1.25 to manufacture and are sold for $2.50. If a bulb burns out before 100 hours the customer is entitled to a full refund. What is the expected profit per bulb?

**Answer**

Profit per Bulb = \(-$1.25 \times P[T < 100]\) + ($2.50 - $1.25) \times P[T > 100]

However,

\[
P[T < 100] = 1 - P[T > 100]
\]

\[
= \int_0^{100} e^{-at} \, dt
\]

\[
= 1 - e^{-100a}
\]

\[
= 1 - e^{-1/2}
\]

Hence,

\[
\text{Profit per Bulb} = $1.25 \times e^{-1/2} - $1.25 \times \left(1 - e^{-1/2}\right)
\]

\[
= $2.50 \times e^{-1/2} - $1.25
\]

\[
= $0.2663
\]
The integral shown below, which is a function of \( t \), will not yield to the usual techniques of freshman calculus. Solve and simplify it using your knowledge of Fourier transform theory. Explain your reasoning.

\[
\frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{\sin(800\pi a) \sin(400\pi (t - a))}{a(t - a)} da =
\]
Rewriting the integral slightly, it is readily recognized as the convolution of two “sinc” functions:

\[
\frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{\sin(800\pi a) \sin(400\pi (t-a))}{a(t-a)} da
\]

\[
= \int_{-\infty}^{\infty} \left[ \frac{\sin(800\pi \tau)}{\pi \tau} \right] \left[ \frac{\sin(400\pi (t - \tau))}{\pi (t - \tau)} \right] d\tau = \frac{\sin(800\pi t)}{\pi t} * \frac{\sin(400\pi t)}{\pi t}.
\]

Note that the use of \(a\) instead of \(\tau\) in the problem statement was just meant to “disguise” the convolution slightly.

Convolution in the time domain corresponds to multiplication in the frequency domain. Using the Fourier transform pair

\[
\frac{\sin(\omega_0 t)}{\pi t} \leftrightarrow u(\omega + \omega_0) - u(\omega - \omega_0).
\]

where \(u(\cdot)\) is the unit step function, and noting that a boxcar shape times a boxcar shape is simply another boxcar, we see that the Fourier transform of the integral is just

\[
[u(\omega + 800\pi) - u(\omega - 800\pi)] \times [u(\omega + 400\pi) - u(\omega - 400\pi)]
\]

\[
= [u(\omega + 400\pi) - u(\omega - 400\pi)].
\]

One could interpret this as the cascade of two ideal lowpass filters. (It’s actually easier to see by drawing graphs.)

Hence, the integral simplifies to just a single sinc function, and the answer is

\[
\frac{\sin(400\pi t)}{\pi t}.
\]

If the student doesn’t exactly recall the Fourier pair used above, they may readily re-derive it if needed from the inverse Fourier transform formula:

\[
\frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} \exp(j\omega t) d\omega = \frac{1}{2\pi} \cdot \frac{\exp(j\omega_0 t)}{jt} \bigg|_{\omega = \omega_0} = \frac{1}{\pi t} \cdot \frac{\exp(j\omega_0 t) - \exp(-j\omega_0 t)}{2j} = \frac{\sin(\omega_0 t)}{\pi t}.
\]
Consider the Boolean function \( Z = a \oplus b \oplus c \oplus d \) (XOR4) which is also called the odd parity function (true if an odd number of inputs are true). Please assume an equal worst-case rise/fall time (\( \gamma = 2 \)). Please use \( C_{inv} = 3C_{fet} \) as the input capacitance of a minimum size inverter, and express all delay in units of \( \tau = R_n C_{inv} \) where \( R_n \) is the equivalent resistance of a minimum size nfet (with gate width and length \( L = L_{min} \) and \( W = W_{min} \) respectively).

a. (3pts) Consider a symmetric mirror form complex gate for this function, where each product term in the switching function of the pullup corresponds to a 1 of \( Z \) (a minterm), and each product term in the switching function of the pulldown represents a 0 of \( Z \) (a maxterm). Assume that each true and complemented literal is available. Give the per input logical effort \( g \) for this gate. You do not need to draw the transistor schematic but you should show how you arrived at your answer.  (Note that there are 8 inputs for this gate).

b. (2pts) Consider a minimum size version of this gate. Give the delay of this gate driving a minimum size inverter (include both effort and parasitic components of delay).
c. (5pts) Consider an implementation of \( Z \) (not necessarily the gate in a.) designed to minimize delay driving a very large load of \( 10,000C_{inv} \). Assume each necessary input is the output of a minimum size inverter. Estimate the worst case delay of this implementation from input inverter to output load. You need not design the entire implementation, but you should sketch the approach.
Consider the Boolean function \( Z = a \oplus b \oplus c \oplus d \) (XOR4) which is also called the odd parity function (true if an odd number of inputs are true). Please assume an equal worst-case rise/fall time (\( \gamma = 2 \)). Please use \( C_{\text{inv}} = 3C_{\text{fet}} \) as the input capacitance of a minimum size inverter, and express all delay in units of \( \tau = R_{\text{inv}} C_{\text{fet}} \) where \( R_{\text{inv}} \) is the equivalent resistance of a minimum size nfet (with gate width and length \( L = L_{\text{min}} \) and \( W = W_{\text{min}} \) respectively).

a. (3pts) Consider a symmetric \textbf{mirror} form complex gate for this function, where each \textbf{product} term in the switching function of the pullup corresponds to a 1 of \( Z \) (a minterm), and each \textbf{product} term in the switching function of the pulldown represents a 0 of \( Z \) (a maxterm). Assume that each true and complemented literal is available. Give the per input logical effort \( g \) for this gate. You do not need to draw the transistor schematic but you should indicate how you arrived at your answer. (Note that there are 8 inputs for this gate).

This gate has 8 groups of 4 series fets in parallel in the pullup and also in the pulldown. The nfets in the prototype are width 4 and the pfets are width 8. Each input literal is a part of 4 terms in the pullup and 4 terms in the pulldown. Thus the total per-input capacitance is \( 4(4 + 8) = 48C_{\text{fet}} \) and the per-input logical effort is given by \( g = 48 / 3 = 16 \).

b. (2pts) Consider a minimum size version of this gate. Give the delay of this gate driving a minimum size inverter (include both effort and parasitic components of delay).

A minimum size XOR4 is scaled to minimum width nfets (and double minimum width pfets and thus has input capacitance \( 12C_{\text{fet}} = 4C_{\text{inv}} \)). We estimate parasitic delay \( p = 8\tau \), effort delay \( f = gh = 16 \cdot 0.25 = 4\tau \) and total delay \( d = f + p = 12\tau \).
c. (5pts) Consider an implementation of $Z$ (not necessarily the gate in a.) designed to minimize delay driving a very large load of $10,000C_{inv}$.
Assume each necessary input is the output of a minimum size inverter.
Estimate the worst case delay of this implementation from input inverter to output load. You need not design the entire implementation, but you should sketch the approach.

Since the electrical effort is so large, we expect a NAND2 implementation to be best. There are 8 4-input AND terms ORed together which looks like INV-NAND-INV-NAND-NAND-INV-NAND-INV-NAND and since each input appears in 4 terms, $F = GBH = (4/3)^5 \cdot 4 \cdot 10,000 = 168,560$ and

$$\hat{N} = \log_{3,6} F = 9.4.$$ Therefore this 9-stage design is the correct one. Then

$$\hat{f} = F^{1/9} = 3.8\tau \quad \text{and} \quad d = \hat{N}\hat{f} + p = (9\cdot 3.8 + 14)\tau = 48.2\tau$$
The illustrated system comprises three synchronous machines. SM1 operates as a generator while machines SM2 and SM3 are a motor generator pair (common shaft, speed of machines SM2 and SM3 are identical). In addition assume that the generated voltage in SM2 and SM3 are equal and the power factor of SM2 is unity. Further neglect losses in the motor/generator set so that the real power input to the motor equals the real power output of the generator. The illustrated circuit is the per-phase equivalent model of the system.

The system operates in steady state conditions. The constant power load absorbs 1.2 MW (one phase) with a power factor of 0.95 (current lagging) while the terminal voltage is 2,400 V, 50 Hz (phase to neutral).

Computed the generated real and reactive power of synchronous machine SM1.

Use a simple synchronous machine model, i.e. constant voltage behind a synchronous impedance. Each of the three synchronous machines has a per phase inductance of 0.0015 Henries.

Hint: Compute the frequency of the currents and voltages in SM1 and SM2.
Solution:

\[ V_L = 2,400 \ e^{0^{\circ}} \]
\[ I_L = 526.32 \ e^{-j18.1949^{\circ}} \]
\[ V_1 = j(0.002)(2050)(526.32 \ e^{j18.1949^{\circ}}) + 2,400 \]
\[ E_3 = j(0.0015)(2050)(526.32 \ e^{j18.1949^{\circ}}) + V_1 \]
\[ E_3 = 2,580.7 \ + j549.78 = 2,638.61 \ e^{j12.0262^{\circ}} \]
\[ E_2 = 2,638.61 \ e^{j12.0262^{\circ}} \]
\[ E_2 \text{ is } 60 \text{ Hz voltage} \]

because of the poles

Motor Generator Set

\[ \text{Re} \left( E_2 \bar{I}_2^* \right) = \text{Re} \left( E_3 \bar{I}_L^* \right) = 1.2 \text{ MW} \]

and

\[ \bar{E}_2 \bar{I}_2^* = 0 \]

\[ \Rightarrow \bar{I}_2 = 4.548 \ A \ e^{j12.0262^{\circ}} \]

\[ V_3 = j(0.0035)(2060)(454.8 \ e^{j12.0262^{\circ}}) + E_2 \]
\[ = 2,705.988 \ e^{j24.834^{\circ}} \]
\[ I_3 = 454.8 \ e^{j12.0262^{\circ}} + 2,705.988 \ e^{j24.834^{\circ}} \]
\[ = 949.838 + j131.65 = 958.958 \ e^{j7.891^{\circ}} \ A \]

\[ S_3 = V_3 \bar{I}_3^* = 2.482 \text{ MW} + j0.756 \text{ MVAR} \]
In the following circuit, the amplifier output resistance is 500 Ω, and \( A(s) \) is given by:

\[
A(s) = \frac{10^7}{s+50}
\]

(a) Find an expression for the loop gain.

(b) What is the maximum load capacitance, \( C_L \), which can be connected to the amplifier if the phase margin is to be at least 50°?

(c) How do you change the amplifier to be able to increase the load capacitance by a factor of 10 while maintaining the 50° phase margin?
(a)  
\[ T(s) = \frac{A\beta}{1 + \frac{sC_L}{R_s}} = \left( \frac{10^7}{s + 50} \right) \frac{1}{sC_L R_s + 1} = \left( \frac{10^7}{s + 50} \right) \frac{1}{500 sC_L + 1} \]

(b) Assume that the unity-gain occurs at \( \omega_i \gg 50 \):  
\[ \angle T(j\omega_i) = \angle A + \angle \beta = -90^\circ - \tan^{-1}\left(\frac{500 \omega_i C_L}{\omega_i}ight) \]  
\[ = -90^\circ - \tan^{-1}\left(\frac{500 \omega_i C_L}{\omega_i}ight) = -180^\circ + 50^\circ \]  
\[ \tan^{-1}(500 \omega_i C_L) = 40^\circ \]  
\[ 500 \omega_i C_L = 0.8391 \]  
\[ \left| T(j\omega) \right| = 1 \rightarrow \frac{10^7}{\omega_i \sqrt{1 + (500 \omega_i C_L)^2}} = \frac{10^7}{\omega_i \sqrt{1 + \left(\tan(40^\circ)\right)^2}} = 1 \rightarrow \omega_i = 7.66 \times 10^6 \]  
\[ C_L = \frac{\tan(40^\circ)}{500 (7.66 \times 10^6)} = 219 \text{ pF} \]

(c) Based on the above equations, if the amplifier output resistance is reduced by a factor of 10, i.e. from 500 \( \Omega \) to 50 \( \Omega \), then \( C_L \) can be increased by a factor of 10 without affecting the phase margin.
PART I: (5 pts)
Consider an electron moving in a hypothetical semiconductor crystal with a lattice constant = 0.730 nm. You may assume that the electron scattering limited velocity is $1 \times 10^8$ cm/s. Give a compelling argument (using numbers) that the proof of the existence of band structure requires solution of the Schrödinger equation in this periodic system. Take $h = 6.634 \times 10^{-34}$ J-s, and $m^* = 9.11 \times 10^{-31}$ kg. **For credit you must carefully show how you got your answer.**

PART II: (5 pts)
Carefully consider two hypothetical diamond lattice semiconductors (Technium and Georgium) and their respective band structures, and then answer the questions below. **For credit you must carefully demonstrate how you got your answers.**

![Band structure graphs](image)

i) Which semiconductor has the **largest** electron mobility?

ii) Which semiconductor has the **smallest** lattice constant?
PART I:
Consider an electron moving in a hypothetical semiconductor crystal with a lattice constant $a = 0.730$ nm. You may assume that the electron scattering limited velocity is $1 \times 10^8$ cm/s.
Give a compelling argument (using numbers) that the proof of the existence of band structure requires solution of the Schrödinger equation in this periodic system. Take $\hbar = 6.634 \times 10^{-34}$ J-s, and $m^* = 9.11 \times 10^{-31}$ kg. For credit you must carefully show how you got your answer.

\[ a = \frac{\hbar}{p} = \frac{6.63 \times 10^{-34} \text{ J-s}}{9.11 \times 10^{-31} \text{ kg} \times 1 \times 10^8 \text{ cm/s}} = 0.73 \text{ nm} \]

\[ \text{e\text{-}wavelength} \gg \text{ lattice constant} \]
\[ \Rightarrow \text{ full QM solution required} \]

PART II:
Carefully consider two hypothetical diamond lattice semiconductors (Technium and Georgium) and their respective band structures, and then answer the questions below. For credit you must carefully demonstrate how you got your answers.

![Technium and Georgium band structures](image)

i) Which semiconductor has the \textbf{largest} electron mobility?

\[ m^* \propto \frac{1}{\frac{d^2E}{dk^2}} \text{ and } m \propto \frac{1}{m^*} \Rightarrow \text{strongest curvature band gap for } E = \text{Technium} \Rightarrow \text{smallest } m^* \Rightarrow \text{largest mobility} \]

\[ \Rightarrow \text{Technium} \]

ii) Which semiconductor has the \textbf{smallest} lattice constant?

\[ E_g \propto \frac{1}{a} \Rightarrow \text{smallest band gap (largest bandgap)} \]
\[ \text{smallest lattice constant} \]
\[ \Rightarrow \text{Technium} \]

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Consider the feedback system shown in Figure 1, where $G(s)$ represents the plant, and the compensation is a proportional controller with gain $K > 0$. The root locus of the system is shown in Figure 2, where only one of the poles is specified. Suppose that the DC gain of the plant is $G(0) = 1$, and the closed-loop system stability range of the gain $K$ is $K < \frac{32}{3}$. Find $G(s)$, and compute the points where the root locus crosses the imaginary axis.

Figure 1: Closed-Loop System

Figure 2: Root Locus
From the root locus we see that

\[ G(s) = \frac{A}{(s + 3)(s + a)^2} \]

for some \( A \in \mathbb{R} \) and \( a > 0 \). The DC-gain information yields that

\[ A = 3a^2. \]

The closed-loop poles are solutions of the equation

\[ s^3 + (2a + 3)s^2 + (a^2 + 6a)s + 3a^2 + KA = 0, \]

and substituting \( A \to 3a^2 \), we get,

\[ s^3 + (2a + 3)s^2 + (a^2 + 6a)s + 3a^2(1 + K) = 0. \]

For \( K = \frac{32}{3} \) there are two imaginary solutions, at \( s = \pm j\omega \) for \( \omega > 0 \). Plug this in the last equation to obtain,

\[ -j\omega^3 + j\omega(a^2 + 6a) - \omega^2(2a + 3) + 3a^2 + 32a^2 = 0. \]

Solve the real and imaginary parts of this equation to obtain,

\[ \omega^2 = a^2 + 6a, \]

and

\[ 2a^2 - 20a + 18 = 0. \]

The solution is at \( a = 1 \) and \( \omega = \sqrt{7} \), hence

\[ G(s) = \frac{3}{(s + 3)(s + 1)^2} \]

and the crossing of the \( j\omega \) axis is at \( \pm j\sqrt{7} \).
TCP

How does a reliable transport protocol (e.g. TCP) overcome the following problems:

1. Lost packet?

2. Bit error in packet?

3. Packets that arrive in wrong order?

4. Packets arrive too fast for receiver to handle.

Internet Delay

Between hosts A in Atlanta and B in Las Dallas there are 2 routers (X,Y). The link between routers (---) is 10 Mbps. The access links (LANs, ===) are 1000 Mbps. The total distance from A to B is 2000 km. The signal travels about 2E8 m/s. A starts to send a large file using TCP, sending 1200 byte packets to B. B acks with 100 byte packets. There is no other traffic on this network.

\[ A ===X---Y===B \]

5. What is the time required to transmit a 1200 Byte datagram at 10 Mbit/s? __________ ms

6. What is the propagation delay for a one-way trip in milliseconds (ms)? __________ ms

7. If the router buffers are empty, what is the total round trip transmission delay (neglecting processing delay and the transmission delay on the X-Y link) Assume the ACK packet is also 1200 bytes? __________ ms

8. Other traffic builds up the average level in X's X-to-Y output buffer to 500 kBytes. What does this queuing delay add to the RTT? __________ ms

9. If a Web Browser (or any network program) wants to find the IP address from a name (e.g., “www.gatech.edu”), to what type of server does it send a query? __________

10. If that server does not have information in its cache, it does a recursive lookup. What type of server does it send its first query to? __________
TCP

How does a reliable transport protocol (e.g. TCP) overcome the following problems:

1. Lost packet? Sender retransmits (when no ACK received)
2. Bit error in packet? Packet will be dropped, and sender retransmits
3. Packets that arrive in wrong order? Sequence number, data to right buffer address
4. Packets arrive too fast for receiver to handle. “Window” size in ACK gets small

Internet Delay

Between hosts A in Atlanta and B in Dallas there are 2 routers (X,Y). The link between routers (---) is 10 Mbps. The access links (LANs, ====) are 1000 Mbps. The total distance from A to B is 2000 km. The signal travels about 2E8 m/s. A starts to send a large file using TCP, sending 1200 byte packets to B. B acks with 100 byte packets. There is no other traffic on this network.

A ====X---Y====B

5. What is the time required to transmit a 1200 Byte datagram at 10 Mbit/s? 0.96 _____ ms

8 * 1200 / 10,000,000 = 9.6e-4 s

6. What is the propagation delay for the one-way trip in milliseconds (ms)? 10.0 _____ ms

2000 km / 2E8 m/s = 1E-2 s = 10 ms

7. If the router buffers are empty, what is the total round trip transmission delay (neglecting processing delay and the transmission delay on the X-Y link)? Assume the ACK packet is also 1200 bytes?

4 * 0.96 + 2 * 10 = 23.84 ms

24 _____ ms

8. Other traffic builds up the average level in X’s X-to-Y output buffer to 500 kBytes. What does this queuing delay add to the RTT?

8 * 500,000 / 1000E6 = 4E6 / 1E7 = 0.4 s = 400 ms

400 _____ ms

9. If a Web Browser (or any network program) wants to find the IP address from a name (e.g., “www.gatech.edu”), to what type of server does it send a query?

(Local) DNS

10. If that server does not have information in its cache, it does a recursive lookup. What type of server does it send its first query to?

Root DNS
A lossless transmission line uses a dielectric insulating material with $\varepsilon_r = 4$. If the line capacitance is $C' = 10 \text{ pF/m}$, find:

(a) The phase velocity $u_p$
(b) The line inductance $L'$, and
(c) The characteristic impedance $Z_0$
(a) 
\[ u_p = \frac{c}{\sqrt{\varepsilon_r}} = \frac{3 \times 10^8}{\sqrt{4}} = 1.5 \times 10^8 \text{ m/s}. \]

(b) 
\[ u_p = \frac{1}{\sqrt{\mu_r}}, \quad u_p^2 = \frac{1}{\mu_0}. \]
\[ L' = \frac{1}{\mu_0 C^2} = \frac{1}{(1.5 \times 10^8)^2 \times 10 \times 10^{-12}} = 4.45 \text{ (}\mu\text{H/m)}. \]

(c) 
\[ Z_0 = \sqrt{\frac{L'}{C^2}} = \left( \frac{4.45 \times 10^{-6}}{10 \times 10^{-12}} \right)^{1/2} = 667.1 \text{ } \Omega. \]
(a) An imaging system consists of two thin lenses. The focal length of the first lens is 4 cm, and the focal length of the second lens is 2.5 cm. The second lens is placed 10 cm behind the first lens. An object is placed 5 cm in front of the first lens. Find the location of the image formed by the system, relative to the position of the second lens. You may use the paraxial approximation.

(b) A coating is applied to the front surface of the first lens to minimize reflections. The first lens has a refractive index of 1.5, and the coating has a refractive index of 1.23. Calculate the coating thickness that will minimize the reflected power for normally incident light with a free-space wavelength of 500 nm.
(a) An imaging system consists of two thin lenses. The focal length of the first lens is 4 cm, and the focal length of the second lens is 2.5 cm. The second lens is placed 10 cm behind the first lens. An object is placed 5 cm in front of the first lens. Find the location of the image formed by the system, relative to the position of the second lens. You may use the paraxial approximation.

\[
\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}
\]

lens 1: \( \frac{1}{5} + \frac{1}{s'} = \frac{1}{4} \) \( \Rightarrow s' = 20 \text{ cm} \)

lens 2: 
\[
-\frac{1}{10} + \frac{1}{5} = 0.25
\]
\[
-\frac{1}{10} + \frac{1}{5} = \frac{2}{5} \Rightarrow s' = 2 \text{ cm}
\]

image is 2 cm after 2nd lens

(b) A coating is applied to the front surface of the first lens to minimize reflections. The first lens has a refractive index of 1.5, and the coating has a refractive index of 1.23. Calculate the coating thickness that will minimize the reflected power for normally incident light with a free-space wavelength of 500 nm.

Should be quarter-wave thick, or odd multiple

\[
t = \frac{\lambda}{4} = \frac{\lambda_0}{n^2} = \frac{500 \text{ nm}}{4 (1.23)}
\]

\[ t = 102 \text{ nm} \]
Consider a 1x4 fiber optic splitter shown here, in which input power through the single mode fiber at the left, $P_{in}$, is (ideally) split equally into the four fibers at the right. The device works in a similar manner in reverse, in which power that is input in any of the four fibers at the right leaves through the left fiber, but with a power reduction of one-fourth (6dB). The loss in decibels (dB) is defined as $10 \log_{10} \left( \frac{P_{out}}{P_{in}} \right)$. A power quantity in decibels relative to a milliwatt is defined as $P_{[dBm]} = 10 \log_{10} \left( \frac{P_{[mW]}}{1mW} \right)$.

\[
\begin{array}{c}
\text{P}_{\text{in}} \\
\rightarrow \\
\text{P}_1 \\
\rightarrow \\
\text{P}_2 \\
\rightarrow \\
\text{P}_3 \\
\rightarrow \\
\text{P}_4 \\
\end{array}
\]

a. Suppose an input power, $P_{in} = 10$ dBm is used, and the device behaves ideally as described above. What dBm value will each of the output ports carry, and thus how many mW of power is found at each port?

b. In a real device, the sum of all four output powers is likely to be less than the input power. The power that is lost is the insertion loss of the device, expressed in decibels through

\[ L_{\text{ins}} = 10 \log_{10} \left( \frac{\sum_{i=1}^{4} P_i}{P_{in}} \right) \]

Suppose that the insertion loss for a given 1x4 splitter is quoted as 4dB, and the output powers are otherwise all equal. What is the power loss in dB between $P_{in}$ and $P_2$, for example? Be careful here, and clearly show your reasoning for full credit.

c. It is possible to construct a 4x4 star coupler, using two 1x4 splitters arranged front-to-front as shown below.

Briefly describe what this device does when power is input to any of the four left ports.

d. Assuming a 4-dB insertion loss in each of the two devices, what is the net dB loss in transmitting, for example, between input port 1 and output port 2?

(continued next page)
e. There are better ways of implementing a star coupler. One involves using four 3dB directional couplers (one of which is shown below). In each coupler, light input in either of the two input ports at left is equally divided between the two output ports at the right. Make a sketch of how four of these can be connected to make a 4x4 star. Also, calculate the net loss in dB for light that transmits through your arrangement, assuming zero insertion loss for the couplers.
Consider a 1x4 fiber optic splitter shown here, in which input power through the single mode fiber at the left, $P_{in}$, is (ideally) split equally into the four fibers at the right. The device works in a similar manner in reverse, in which power that is input in any of the four fibers at the right leaves through the left fiber, but with a power reduction of one-fourth (6dB). The loss in decibels (dB) is defined as $10 \log_{10} (P_{out}/P_{in})$. A power quantity in decibels relative to a milliwatt is defined as $P_{[dBm]} = 10 \log_{10} (P_{[mW]}/1mW)$.

**Solution**

**a.** Suppose an input power, $P_{in} = 10$ dBm is used, and the device behaves ideally as described above. What dBm value will each of the output ports carry, and thus how many mW of power is found at each port?

The power at each port will be $P_i = 10$dBm – 6dB = 4dBm, which translates to mW as 

$$P_i[mW] = 10^{0.4} = 2.5\, mW$$

**b.** In a real device, the sum of all four output powers is likely to be less than the input power. The power that is lost is the **insertion loss** of the device, expressed in decibels through

$$L_{ins} = 10 \log_{10} \left[ \frac{\sum_{i=1}^{4} P_i}{P_{in}} \right]$$

Suppose that the insertion loss for a given 1x4 splitter is quoted as 4dB, and the output powers are otherwise all equal. What is the power loss in dB between $P_{in}$ and $P_2$, for example? Be careful here. 

This will simply be $4$dB (same for all ports).

**c.** It is possible to construct a 4x4 **star coupler**, using two 1x4 splitters arranged front-to-front as shown below.

Briefly describe what this device does when power is input to any of the four left ports.

*Power from any left port couples into the center line with (ideally) a 6dB loss, whereupon it enters the second splitter, which divides the power equally into all four outputs. Total loss ideally will be 12 dB.*

**d.** Assuming a 4-dB insertion loss in each of the two devices, what is the net dB loss in transmitting from any left-hand port to any right-hand port?

This will be $2(6+4) = 20$dB (continued next page)
e. There are better ways of implementing a star coupler. One involves using four 3dB directional couplers (one of which is shown below). In each coupler, light input in either of the two input ports at left is equally divided between the two output ports at the right. Make a sketch of how four of these can be connected to make a 4x4 star. Also, calculate the net loss in dB for light that transmits through your arrangement, assuming zero insertion loss for the couplers.

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and with no insertion loss, the throughput loss will be just 6dB (2x3).
Problem 21 (Specialized: BIO - ECE 4782) Solution

PROBLEM

(4 pts) Cardiovascular Anatomy / Physiology. Fill in the blanks with the best possible answer.

The two large veins which bring deoxygenated blood to the right atrium from most of the body are the _______________ and _______________.

The volumetric flow rate of blood from the heart is called _______________ and is the produce of heart rate and _______________.

(4 pts) Draw an electrocardiogram (ECG) waveform for one heartbeat, and label the P and T waves, the QRS complex, and the approximate amplitude of the QRS complex.

(2 pts) If you are designing an algorithm for classifying premature atrial contractions (an arrhythmia related to hypoxic foci in the atria), provide a mathematical expression relating the R-to-R interval that precedes (RR<sub>i-1</sub>) and follows (RR<sub>i+1</sub>) the arrhythmia to the average interval for the entire recording (∑<sub>j=1</sub>^∞ RR<sub>j</sub>), assuming the rest of the recording is composed of only normal heartbeats.
(4 pts) **Cardiovascular Anatomy / Physiology.** Fill in the blanks with the *best* possible answer.

The two large veins which bring deoxygenated blood to the right atrium from most of the body are the __superior vena cava__ and __inferior vena cava__.

The volumetric flow rate of blood from the heart is called __cardiac output__ and is the produce of heart rate and ___stroke volume__.

(4 pts) Draw an electrocardiogram (ECG) waveform for one heartbeat, and label the P and T waves, the QRS complex, and the approximate amplitude of the QRS complex.

The student will get 1 point for drawing the ECG, 1 point for labeling the P and T waves correctly, 1 point for labeling the QRS complex, and 1 point for providing an amplitude of approximately 1 mV.

(2 pts) If you are designing an algorithm for classifying premature atrial contractions (an arrhythmia related to hypoxic foci in the atria), provide a mathematical expression relating the R-to-R interval that precedes (RR\(_i-1\)) and follows (RR\(_i+1\)) the arrhythmia to the average interval for the entire recording (RR\(_\bar{a}v\)), assuming the rest of the recording is composed of only normal heartbeats.

\[
RR_i-1 < RR_{avg} \\
RR_i+1 > RR_{avg}
\]
Problem 22 (Specialized: BIO - ECE 4784) Solution

**Problem**

(3 points) **Electrochemical Balance.** For this question, we are concerned with a cell (T=20° C) with the initial ionic concentrations listed below. \( A^- \) represents arbitrary anions. The membrane is passive and only permeable to \( K^+ \) and \( Cl^- \) ions.

<table>
<thead>
<tr>
<th>Ion</th>
<th>Intracellular (mM)</th>
<th>Extracellular (mM)</th>
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</thead>
<tbody>
<tr>
<td>( K^+ )</td>
<td>145</td>
<td>145</td>
</tr>
<tr>
<td>( Na^+ )</td>
<td>10</td>
<td>100</td>
</tr>
<tr>
<td>( Cl^- )</td>
<td>55</td>
<td>245</td>
</tr>
<tr>
<td>( A^- )</td>
<td>100</td>
<td>0</td>
</tr>
</tbody>
</table>

Calculate 1) the final concentration of extracellular \( K^+ \) and 2) the resulting membrane potential at equilibrium. Show all work and specify units.
(4 points) **Current-voltage relationships.**

The figure below illustrates 4 different steady-state current-voltage (I-V) plots. Each I-V plot represents a combination of one or more membrane ion channels: Na\(^+\), Cl\(^-\), and/or K\(^+\). Each ion channel may be linear, or have a voltage-activated or voltage-inactivated conductance. The Nernst potential for the 3 ionic species are \(E_K = -90\) mV, \(E_{Cl} = -55\) mV, and \(E_{Na} = +50\) mV.

Determine a combination of ionic current(s) sufficient to generate each I-V curve. For each current in your answer, specify both the ionic species and whether the conductance is constant, voltage-activated, or voltage-inactivated. Each answer should have an explanation justifying your answer with equations or reasoning. You are also free to draw on the graphs. Note: it is possible that an I-V plot may have more than one correct answer, and credit will be given for any correct answer.
(3 points) **Nerve Stimulation.** A strength duration curve for a nerve is below. Determine parameters (amplitude, pulse duration) for a periodic stimulus protocol that will maximize battery life for an implantable stimulator. The maximum safe charge injection (total charge to be injected over a single pulse) is 0.15 $\mu$C. The stimulator circuitry cannot source a current larger than 10 mA. Assume a monophasic pulse for your analysis (this is not realistic, but simplifies assumptions). Justify your answer.
(3 points) **Electrochemical Balance.** For this question, we are concerned with a cell (T=20° C) with the initial ionic concentrations listed below. A⁻ represents arbitrary anions. The membrane is passive and only permeable to K⁺ and Cl⁻ ions.

<table>
<thead>
<tr>
<th>Ion</th>
<th>Intracellular (mM)</th>
<th>Extracellular (mM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>K⁺</td>
<td>145</td>
<td>145</td>
</tr>
<tr>
<td>Na⁺</td>
<td>10</td>
<td>100</td>
</tr>
<tr>
<td>Cl⁻</td>
<td>55</td>
<td>245</td>
</tr>
<tr>
<td>A⁻</td>
<td>100</td>
<td>0</td>
</tr>
</tbody>
</table>

Calculate 1) the final concentration of extracellular K⁺ and 2) the resulting membrane potential at equilibrium. Show all work and specify units.

The system is not in equilibrium. Of the permeable ions, K⁺ is in equilibrium, but Cl⁻ is not. Donnan equilibrium applies. Cl⁻ will flow from high (extracellular) to low (intracellular concentration). For each Cl⁻ ion that flows, a K⁺ ion will flow in the same direction to maintain charge neutrality. At equilibrium, the Nernst potentials will be the same. Note the minus sign in $E_{Cl}$ below due to the negative valence.

\[
E_K = \frac{RT}{F} \ln \left( \frac{[K^+]_{\text{out}}}{[K^+]_{\text{in}}} \right)
\]

\[
E_{Cl} = -\frac{RT}{F} \ln \left( \frac{[Cl^-]_{\text{out}}}{[Cl^-]_{\text{in}}} \right)
\]

\[
\frac{[K^+]_{\text{out}}}{[K^+]_{\text{in}}} = \frac{[Cl^-]_{\text{in}}}{[Cl^-]_{\text{out}}}
\]

Solving for $x$ ...

\[
\frac{145-x}{145+x} = \frac{55+x}{245-x}
\]

$x$ is approximately 47 mM. Therefore, $[K^+]_{\text{out}} = 145-x = 98$ mM. The membrane potential will be the Nernst potential of either ion, so $E_K = 0.025 \ln \frac{145-x}{145+47} = -17$ mV.
(4 points) **Current-voltage relationships.**

The figure below illustrates 4 different steady-state current-voltage (I-V) plots. Each I-V plot represents a combination of one or more membrane ion channels: Na\(^{+}\), Cl\(^{-}\), and/or K\(^{+}\). Each ion channel may be linear, or have a voltage-activated or voltage-inactivated conductance. The Nernst potential for the 3 ionic species are \(E_{K} = -90\) mV, \(E_{Cl} = -55\) mV, and \(E_{Na} = +50\) mV.

A: I-V curve passes through \(E_{K}\), so at low potentials there is a \(K^{+}\) current that is fully activated. No other currents present or necessary. As the voltage increases, a **voltage-inactivating \(K^{+}\) current** would generate that I-V curve.

B: The lower zero crossing is above \(E_{K}\) and less than \(E_{Cl}\), so assume both \(K^{+}\) and either \(Na^{+}\) or \(Cl^{-}\) currents are present. The higher zero crossing appears to be at \(E_{Cl}\), thus at that voltage only \(Cl^{-}\) ions are flowing. Thus we can assume that \(K^{+}\) and \(Cl^{-}\) ions are flowing at lower potentials, but by -55 mV only \(Cl^{-}\) ions are flowing. The simplest explanation is a **voltage-inactivating \(K^{+}\) current and a linear \(Cl^{-}\) current**. This is further confirmed by the fact that the slow at the lower potentials is greater, implying a higher conductance (both ion channels fully open) vs at higher potentials (just \(Cl^{-}\)).

C: At higher potentials the I-V curve is linear and crosses at \(E_{Na}\), thus there is an \(Na^{+}\) current that is fully open. At lower potentials the slope increases, implying another current is active in addition to \(Na^{+}\), thus there is a linear \(Na^{+}\) current. That additional current is negative above \(E_{K}\), thus it cannot be a \(K^{+}\) current and is a \(Cl^{-}\) current that is active at lower potentials but turns off at higher potentials. Thus the curve represents a **voltage-inactivating \(Cl^{-}\) current and a linear \(Na^{+}\) current**.

D: The I-V curve is linear, thus a sum of more than one linear current. The reversal potential is above \(E_{K}\) and \(E_{Cl}\) and below \(E_{Na}\). Thus a **combination of a linear \(Na^{+}\) current and either a linear \(Cl^{-}\) or \(K^{+}\) current** could generate that curve.
Problem 22 (Specialized: BIO - ECE 4784) Solution

(3 points) Nerve Stimulation. A strength duration curve for a nerve is below. Determine parameters (amplitude, pulse duration) for a periodic stimulus protocol that will maximize battery life for an implantable stimulator. The maximum safe charge injection (total charge to be injected over a single pulse) is 0.15 \( \mu \text{C} \). The stimulator circuitry cannot source a current larger than 10 mA. Assume a monophasic pulse for your analysis (this is not realistic, but simplifies assumptions). Justify your answer.

![Strength Duration Curve](image)

Important for solving this problem is the recognition of what a strength-duration curve is and knowing that the stimuli needed will lie along this curve. The solution with the least amount of charge transfer per cycle and satisfying the other constraints (charge transfer, maximum current) are the most desirable. Looking at the SD curve, 10 mA is about 15 \( \mu \text{sec} \), or about 0.15 \( \mu \text{C} \) of charge transfer, which is acceptable. 6 mA is about 25 \( \mu \text{sec} \), also 0.15 \( \mu \text{C} \) of charge transfer. 4 mA is clearly (no estimate required) at 40 \( \mu \text{sec} \), or 0.160 \( \mu \text{C} \) – too much charge. The charge transfer continues to increase for longer duration stimuli (e.g. 1.75 at 100 \( \mu \text{sec} \)).

Thus points on the SD curve with (duration, amplitude) from (25 \( \mu \text{sec} \), 6 mA) to (15 \( \mu \text{A} \), 10 mA) seem to have the lowest charge transfer and satisfy the maximum current and maximum safe charge transfer constraints. The justification of one particular answer depends on how the student interpolates. Multiple answers are valid depending on the interpolation and the ensuing justification. For example, one might argue that 10 mA is a duration of less than 15 \( \mu \text{sec} \), this the lowest charge transfer, and thus one should use (14 \( \mu \text{sec} \), 10 mA). Alternatively, if one assumes that whole range (10mA to 6mA) is all about the same in terms of charge transfer, one could also argue for (25 \( \mu \text{sec} \), 6 mA) on the ground that all things being equal, a lower amplitude is preferable. Either answer is OK with sufficient justification.