EE6604
Personal & Mobile Communications

Week 5

Level Crossing Rate and Average Fade Duration

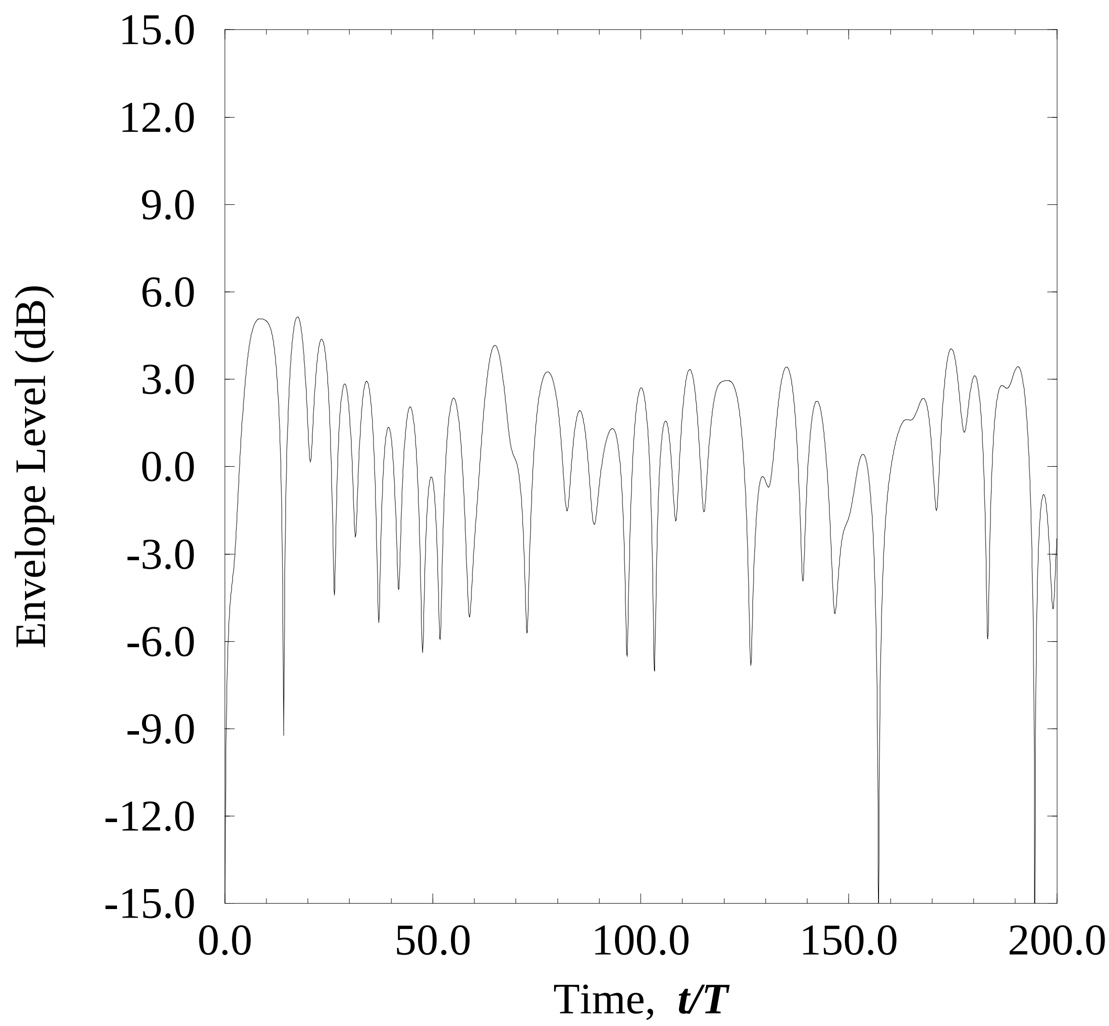
Statistical Channel Modeling

Fading Simulators

Reading: Chapter 2, 2.1.5, 2.3, 2.5.1, 2.5.2

Level Crossing Rate and Average Fade Duration

- The **level crossing rate** (LCR) at a specified envelope level R, L_R, is defined as the rate (in crossings per second) at which the envelope α crosses the level R in the positive going direction.
  - The LCR can be used to estimate velocity, and velocity can be used for radio resource management.
- The **average fade duration** (AFD) is the average duration that the envelope remains below a specified level R.
  - An outage occurs when the envelope fades below a critical level for a long enough period such that receiver synchronization is lost. Longer fades are usually the problem.
  - The probability distribution of fade durations, if it exists, would allow us to calculate probability of outage.
- Both the LCR and AFD are second-order statistics that depend on the mobile station velocity, as well as the scattering environment.
- The LCR and AFD have been derived by Rice (1948) in the context of a sinusoid in narrow-band Gaussian noise.



Rayleigh faded envelope with 2-D isotropic scattering.

Level Crossing Rate

- Obtaining the level crossing rate requires the joint pdf, $p(\alpha, \dot{\alpha})$, of the envelope level $\alpha = |g(t_1)|$ and the envelope slope $\dot{\alpha} = d|g(t_1)|/dt$ at any time instant t_1. *Note we drop the time index t for convenience.*

- In terms of $p(\alpha, \dot{\alpha})$, the expected amount of time the envelope lies in the interval $(R, R+d\alpha)$ for a given envelope slope $\dot{\alpha}$ and time increment dt is

$$p(R, \dot{\alpha})d\alpha d\dot{\alpha}dt$$

- The time required for the envelope α to traverse the interval $(R, R+d\alpha)$ "once" for a given envelope slope $\dot{\alpha}$ is

$$d\alpha/\dot{\alpha}$$

- The ratio of the above two quantities is the expected number of crossings of the envelope α within the interval $(R, R+d\alpha)$ for a given envelope slope $\dot{\alpha}$ and time duration dt, i.e.,

$$\dot{\alpha}p(R, \dot{\alpha})d\dot{\alpha}dt$$

- The expected number of crossings of the envelope level R for a given envelope slope $\dot{\alpha}$ in a time interval of duration T is

$$\int_0^T \dot{\alpha} p(R, \dot{\alpha}) d\dot{\alpha} dt = \dot{\alpha} p(R, \dot{\alpha}) d\dot{\alpha} T$$

- The expected number of crossings of the envelope level R with a positive slope in the time interval T is

$$N_R = T \int_0^\infty \dot{\alpha} p(R, \dot{\alpha}) d\dot{\alpha} \ .$$

- Finally, the expected number of crossings of the envelope level R per second, or the level crossing rate, is obtained by dividing by the length of the interval T as

$$L_R = \int_0^\infty \dot{\alpha} p(R, \dot{\alpha}) d\dot{\alpha}$$

- This is a general result that applies to any random process characterized by the joint pdf $p(\alpha, \dot{\alpha})$.

- Rice (BSTJ, 1948) derived the joint pdf $p(\alpha, \dot{\alpha})$ for a sine wave plus Gaussian noise. A Rician fading channel can be thought of LoS or specular (sine wave) component plus a scatter (Gaussian noise) component. For the case of a Rician fading channel,

$$\begin{aligned} p(\alpha, \dot{\alpha}) &= \frac{\alpha (2\pi)^{-3/2}}{\sqrt{B b_0}} \int_{-\pi}^{\pi} d\theta \\ &\quad \times \exp\left\{ -\frac{1}{2 B b_0} \left[B \left(\alpha^2 - 2\alpha s \cos\theta + s^2 \right) + \left(b_0 \dot{\alpha} + b_1 s \sin\theta \right)^2 \right] \right\} \end{aligned}$$

  where s is the non-centrality parameter in the Rice distribution, and $B = b_0 b_2 - b_1^2$, where b_0, b_1, and b_2 are constants that depend on the scattering environment.

- Suppose that the specular or LoS component of the complex envelope $g(t)$ has a Doppler frequency equal $f_q = f_m \cos\theta_0$, where $0 \le |f_q| \le f_m$. Then

$$\begin{aligned} b_n &= (2\pi)^n \int_{-f_m}^{f_m} S_{gg}^c(f)(f - f_q)^n df \\ &= (2\pi)^n b_0 \int_0^{2\pi} \hat{p}(\theta) G(\theta) \left(f_m \cos\theta - f_q \right)^n d\theta \end{aligned}$$

  where $\hat{p}(\theta)$ is the azimuth distribution (pdf) of the *scatter* component, $G(\theta)$ is the antenna gain pattern, and $S_{gg}^c(f)$ is the corresponding continuous portion of the Doppler power spectrum.

  - Note that the pdf $\hat{p}(\theta)$ in this case integrates to unity.

- Note that $S^c_{gg}(f)$ is given by the Fourier transform of $\phi^c_{gg}(\tau) = \phi^c_{g_Ig_I}(\tau) + j\phi^c_{g_Ig_Q}(\tau)$ where

$$\begin{aligned}
\phi^c_{g_Ig_I}(\tau) &= \frac{\Omega_p}{2}\int_0^{2\pi}\cos(2\pi f_m\tau\cos\theta)\hat{p}(\theta)G(\theta)d\theta \\
\phi^c_{g_Ig_Q}(\tau) &= \frac{\Omega_p}{2}\int_0^{2\pi}\sin(2\pi f_m\tau\cos\theta)\hat{p}(\theta)G(\theta)d\theta
\end{aligned}$$

- In some special cases, the psd $S^c_{gg}(f)$ is symmetrical about the frequency $f_q = f_m\cos\theta_0$. This condition occurs, for example, when $f_q = 0$ ($\theta_0 = 90^{\rm o}$) and $\hat{p}(\theta) = 1/(2\pi), -\pi \leq \theta \leq \pi$.
  - Specular component arrives perpendicular to direction of motion and scatter component is characterized by 2-D isotropic scattering.
  - In this case, $b_n = 0$ for all odd values of n (and in particular $b_1 = 0$) so that the joint pdf $p(\alpha, \dot{\alpha})$ reduces to the convenient product form

$$\begin{aligned}
p(\alpha,\dot{\alpha}) &= \sqrt{\frac{1}{2\pi b_2}}\exp\left\{-\frac{\dot{\alpha}^2}{2b_2}\right\}\cdot\frac{\alpha}{b_0}\exp\left\{-\frac{(\alpha^2+s^2)}{2b_0}\right\}I_0\left(\frac{\alpha s}{b_0}\right) \\
&= p(\dot{\alpha})\cdot p(\alpha)\ .
\end{aligned}$$

  - Since $p(\alpha,\dot{\alpha}) = p(\dot{\alpha})\cdot p(\alpha)$, it follows that α and $\dot{\alpha}$ are independent *for this special case.*

- When $f_q = 0$ and $\hat{p}(\theta) = 1/(2\pi)$, a closed form expression can be obtained for the envelope level crossing rate.
- We have that

$$b_n = \begin{cases} b_0(2\pi f_m)^n \frac{1\cdot 3\cdot 5\ \cdots\ (n-1)}{2\cdot 4\cdot 6\ \cdots\ n} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} .$$

- Therefore, $b_1 = 0$ and $b_2 = b_0(2\pi f_m)^2/2$, and

$$L(R) = \sqrt{2\pi(K+1)} f_m \rho e^{-K-(K+1)\rho^2} I_0\left(2\rho\sqrt{K(K+1)}\right)$$

where

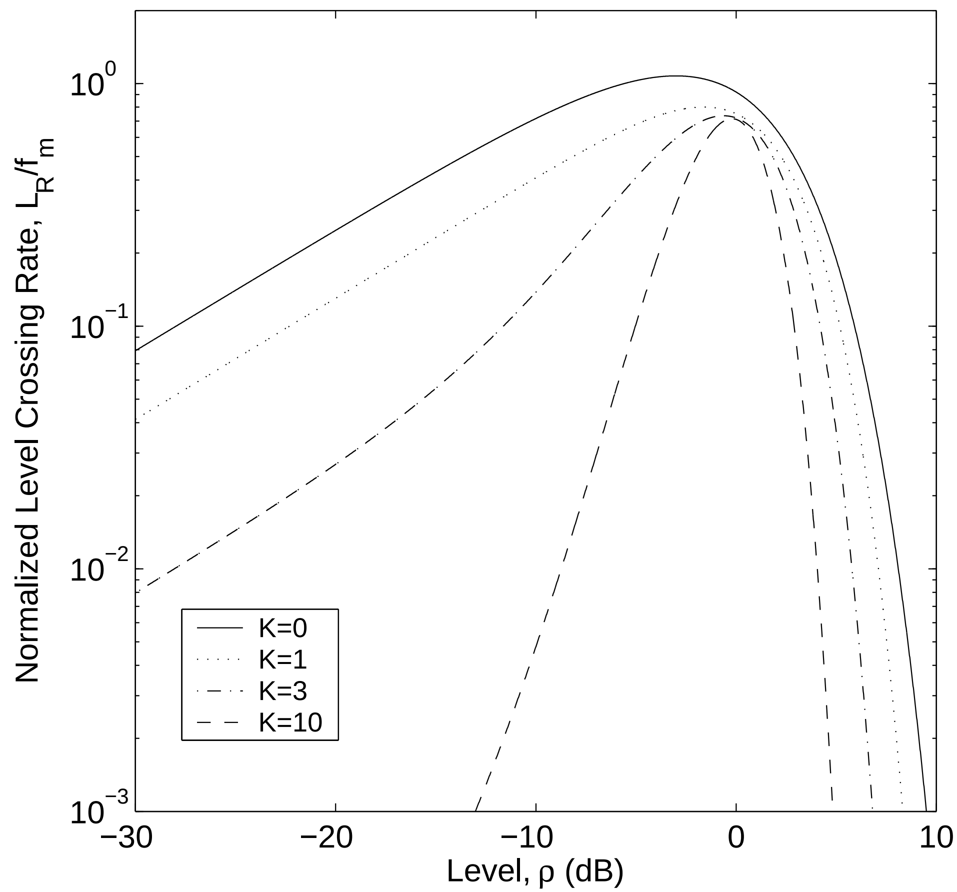
$$\rho = \frac{R}{\sqrt{\Omega_p}} = \frac{R}{R_{\text{rms}}}$$

and $R_{\text{rms}} \triangleq \sqrt{\mathrm{E}[\alpha^2]}$ is the *rms* envelope level.

- Under the further condition that $K = 0$ (Rayleigh fading)

$$L(R) = \sqrt{2\pi} f_m \rho e^{-\rho^2}$$

- Notice that the level crossing rate is directly proportional to the maximum Doppler frequency f_m and, hence, the MS speed $v = f_m \lambda_c$.



Normalized level crossing rate for Rician fading. A specular component arrives with angle $\theta_0 = 90^{\mathrm{o}}$ and there is 2-D isotropic scattering of the scatter component.

Average Fade Duration

- No known probability distribution exists for the duration of fades; this is a long standing open problem! Therefore, we consider the "**average fade duration**".
- Consider a very long time interval of length T, and let t_i be the duration of the ith fade below the level R.
- The probability that the received envelope α is less than R is

$$\Pr[\alpha \leq R] = \frac{1}{T} \sum_i t_i$$

- The average fade duration is equal to

$$\begin{aligned}\bar{t} &= \frac{\text{total length of time in duraton } T \text{ that the envelope is below level } R}{\text{average number of crossings in duration } T} \\ &= \frac{\sum_i t_i}{TL(R)} \\ &= \frac{\Pr[\alpha \leq R]}{L(R)}\end{aligned}$$

- If the envelope is Rician distributed, then

$$\Pr[\alpha \leq R] = \int_0^R p(\alpha) d\alpha = 1 - Q\left(\sqrt{2K}, \sqrt{2(K+1)\rho^2}\right)$$

where $Q(a, b)$ is the Marcum Q function.

- If we again assume that $f_q = 0$ and $\hat{p}(\theta) = 1/(2\pi)$, we have

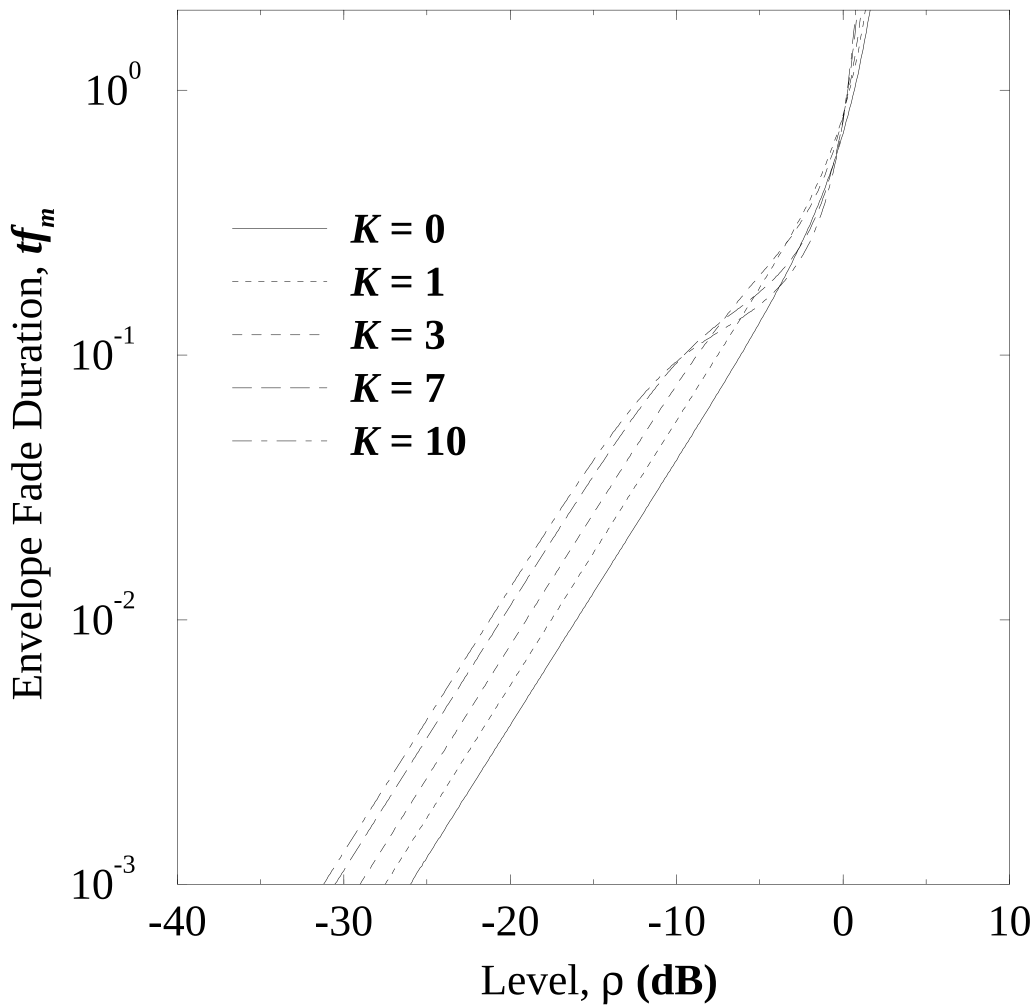
$$\bar{t} = \frac{1 - Q\left(\sqrt{2K}, \sqrt{2(K+1)\rho^2}\right)}{\sqrt{2\pi(K+1)} f_m \rho e^{-K-(K+1)\rho^2} I_0\left(2\rho\sqrt{K(K+1)}\right)}$$

- If we further assume that $K = 0$ (Rayleigh fading), then

$$\mathrm{P}[\alpha \leq R] = \int_0^R p(\alpha) d\alpha \;=\; 1 - e^{-\rho^2}$$

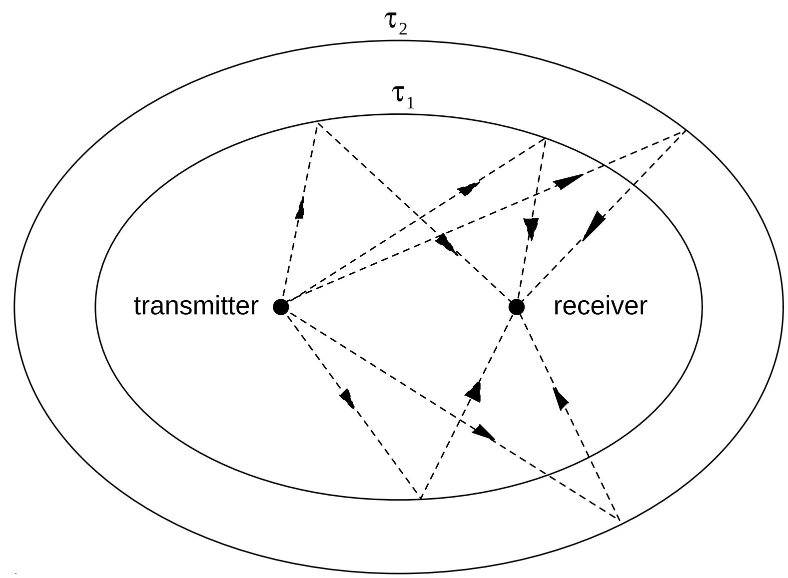
and

$$\bar{t} \;=\; \frac{e^{\rho^2} - 1}{\rho f_m \sqrt{2\pi}} \;.$$



Normalized average fade duration with Ricean fading.

Scattering Mechanism for Wideband Channels



Concentric ellipses model for frequency-selective fading channels.

- Frequency-selective (wide-band) channels have strong scatterers that are located on several ellipses such that the corresponding differential path delays $\tau_i - \tau_j$ for some i, j, are significant compared to the modulated symbol period T.

Transmission Functions

- Multipath fading channels are time-variant linear filters, whose inputs and outputs can be described in the time and frequency domains.
- There are four possible transmission functions
  - Time-variant channel impulse response $g(t, \tau)$
  - Output Doppler spread function $H(f, \nu)$
  - Time-variant transfer function $T(f, t)$
  - Doppler-spread function $S(\tau, \nu)$

Time-variant channel impulse response, $g(t,\tau)$

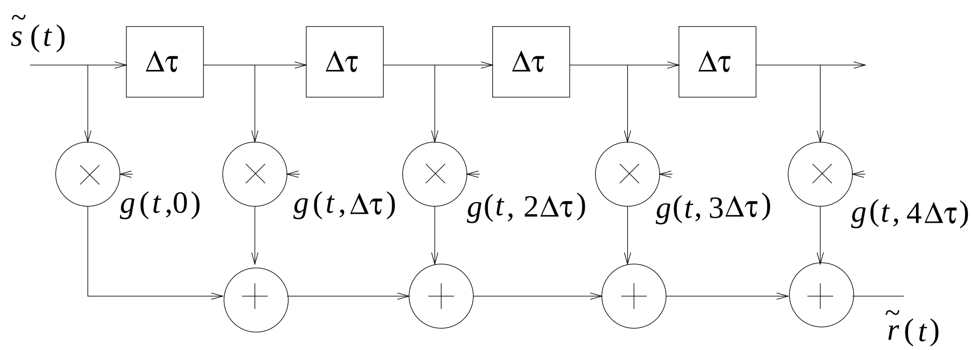
- Also known as the input delay spread function.
- The time varying complex channel impulse response relates the input and output time domain waveforms

$$\tilde{r}(t) = \int_0^t g(t,\tau)\tilde{s}(t-\tau)d\tau$$

  - In physical terms, $g(t,\tau)$ can be interpreted as the channel response at time t due to an impulse applied at time $t-\tau$. Since a physical channel is causal, $g(t,\tau)=0$ for $\tau<0$ and, therefore, the lower limit of integration in the convolution integral is zero.

- The convolution integral can be approximated in the discrete form

$$\tilde{r}(t) = \sum_{m=0}^{n} g(t, m\Delta\tau)\tilde{s}(t-m\Delta\tau)\Delta\tau$$



Discrete-time tapped delay line model for a multipath-fading channel.

Transfer Function, $T(f,t)$

- The transfer function relates the input and output frequencies:

$$\tilde{R}(f) = \tilde{S}(f)T(f,t)$$

- By using an inverse Fourier transform, we can also write

$$\tilde{r}(t) = \int_{-\infty}^{\infty} \tilde{S}(f)T(f,t)e^{j2\pi ft}df$$

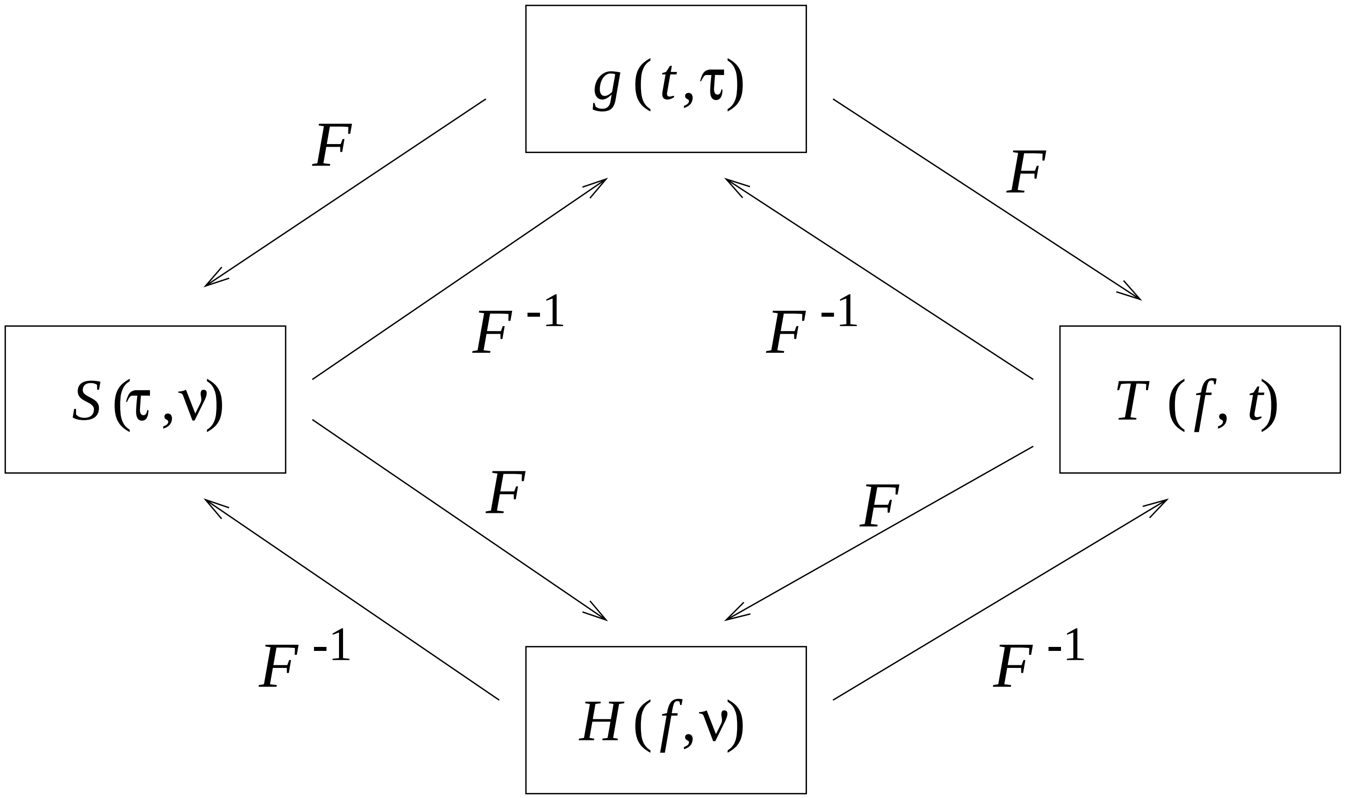
- The time-varying channel impulse response and time-varying channel transfer function are related through the Fourier transform:

$$g(t,\tau) \Longleftrightarrow T(f,t)$$

  – Note: the Fourier transform pair is with respect to the time-delay variable τ. The Fourier transform of $g(t,\tau)$ with respect to the time variable t gives the Doppler spread function $S(\tau,\nu)$, i.e,

$$g(t,\tau) \Longleftrightarrow S(\tau,\nu)$$

Fourier Transforms



Fourier transform relations between the system functions.

Statistical Correlation Functions

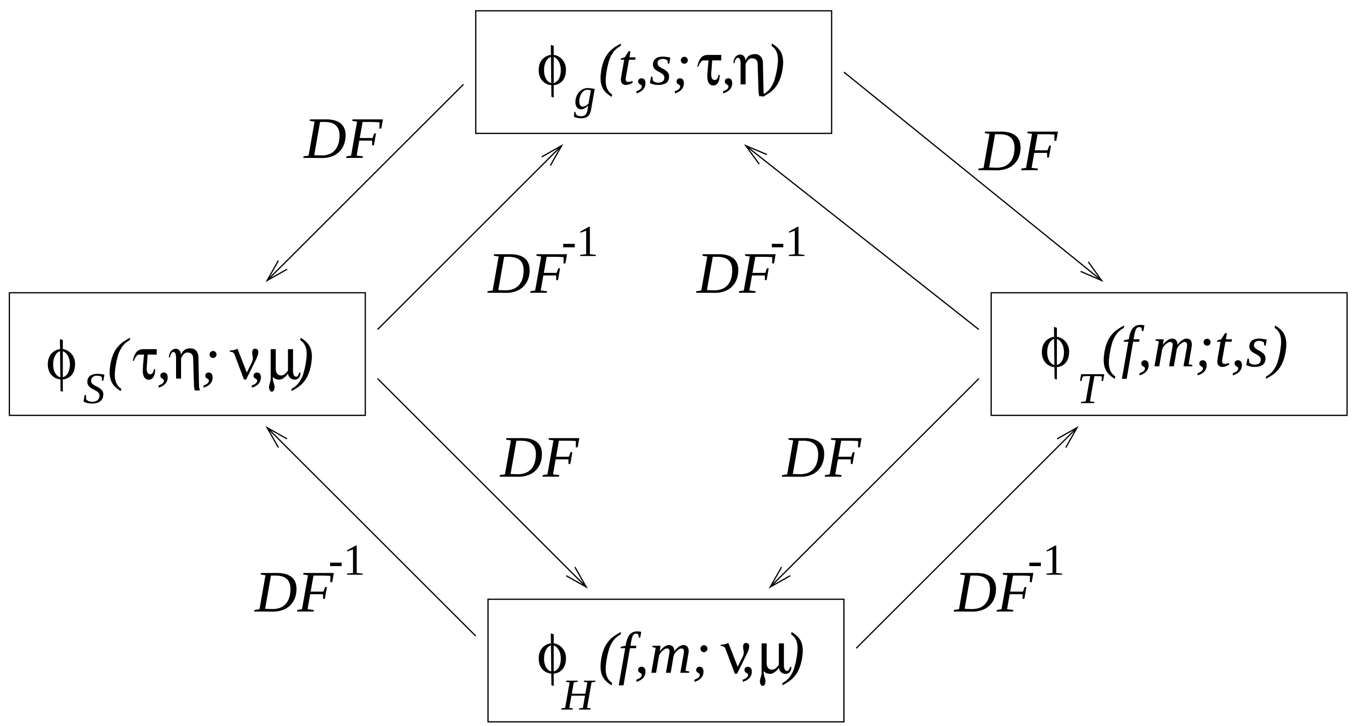
- Similar to flat fading channels, the channel impulse response $g(t,\tau) = g_I(t,\tau) + jg_Q(t,\tau)$ of frequency-selective fading channels can be modelled as a complex Gaussian random process, where the quadrature components $g_I(t,\tau)$ and $g_Q(t,\tau)$ are Gaussian random processes.
- The transmission functions are all random processes. Since the underlying process is Gaussian, a complete statistical description of these transmission functions is provided by their means and autocorrelation functions.
- Four autocorrelation functions can be defined

$$\begin{aligned}
\phi_g(t,s;\tau,\eta) &= \mathrm{E}[g^*(t,\tau)g(s,\eta)] \\
\phi_T(f,m;t,s) &= \mathrm{E}[T^*(f,t)T(m,s)] \\
\phi_H(f,m;\nu,\mu) &= \mathrm{E}[H^*(f,\nu)H(m,\mu)] \\
\phi_S(\tau,\eta;\nu,\mu) &= \mathrm{E}[S^*(\tau,\nu)S(\eta,\mu)] \ .
\end{aligned}$$

- Related through double Fourier transform pairs

$$\begin{aligned}
\phi_S(\tau,\eta;\nu,\mu) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_g(t,s;\tau,\eta)e^{-j2\pi(\nu t-\mu s)}dtds \\
\phi_g(t,s;\tau,\eta) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_S(\tau,\eta;\nu,\mu)e^{j2\pi(\nu t-\mu s)}d\nu d\mu
\end{aligned}$$

Fourier Transforms and Correlation Functions



Double Fourier transform relations between the channel correlation functions.

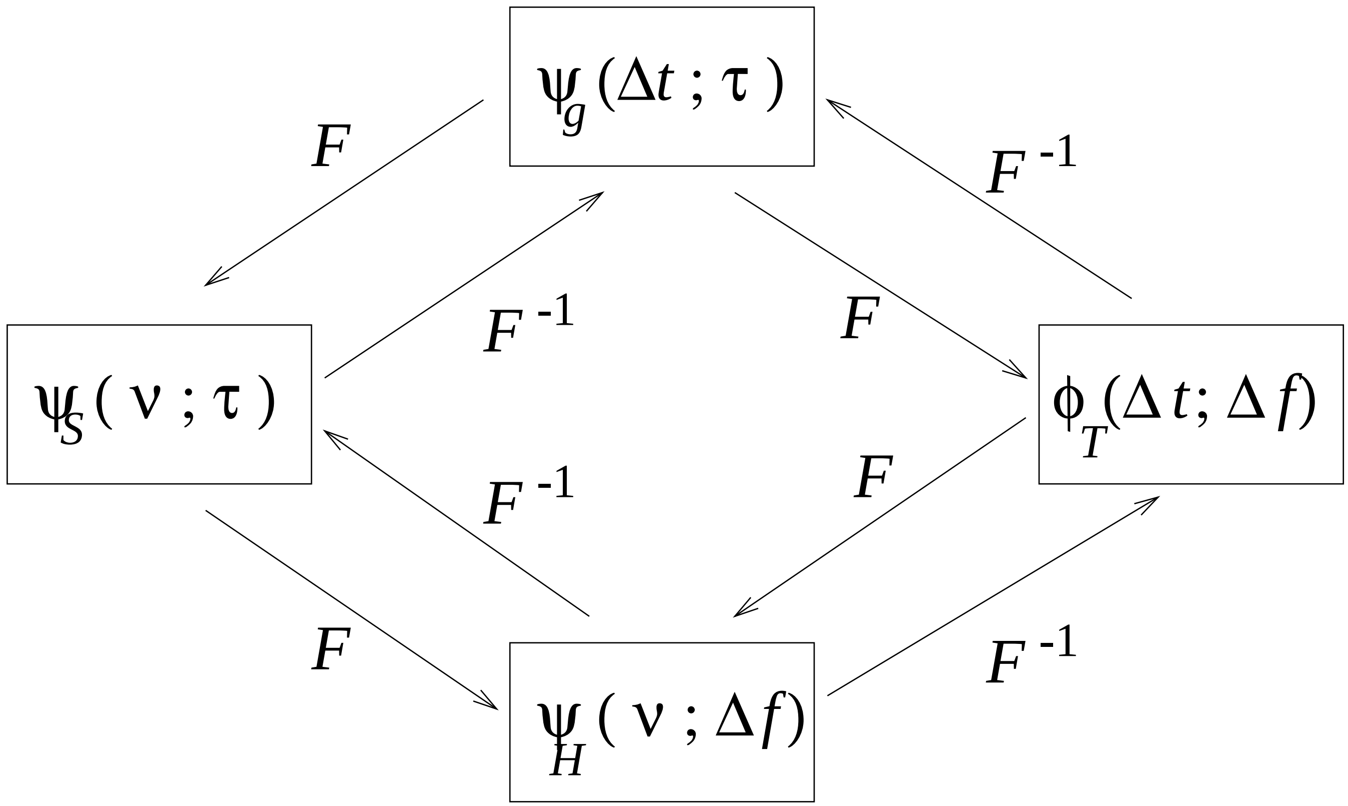
WSSUS Channels

- Uncorrelated scattering in both the time-delay and Doppler shift domains.
- Practical land mobile radio channels are characterized by this behavior.
- Due to uncorrelated scattering in time-delay and Doppler shift, the channel correlation functions become:

$$\begin{aligned}\phi_g(t, t+\Delta t; \tau, \eta) &= \psi_g(\Delta t; \tau)\delta(\eta - \tau)\\ \phi_T(f, f+\Delta f; t, t+\Delta t) &= \phi_T(\Delta f; \Delta t)\\ \phi_H(f, f+\Delta f; \nu, \mu) &= \psi_H(\Delta f; \nu)\delta(\nu - \mu)\\ \phi_S(\tau, \eta; \nu, \mu) &= \psi_S(\tau, \nu)\delta(\eta - \tau)\delta(\nu - \mu)\ .\end{aligned}$$

- Note the singularities $\delta(\eta - \tau)$ and $\delta(\nu - \mu)$ with respect to the time-delay and Doppler shift variables, respectively.
- Some correlation functions are more useful than others. The most useful functions:
  - $\psi_g(\Delta t; \tau)$: **channel correlation function**
  - $\phi_T(\Delta f; \Delta t)$: **spaced-time spaced-frequency correlation function**
  - $\psi_S(\tau, \nu)$: **scattering function**

Fourier Transforms for WSSUS Channels



Power Delay Profile

- The autocorrelation function of the time varying impulse response is

$$\begin{aligned}\phi_g(t, t+\Delta t, \tau, \eta) &= \mathrm{E}\left[g^*(t,\tau)g(t+\Delta t, \eta)\right] \\ &= \psi_g(\Delta t; \tau)\delta(\eta - \tau)\end{aligned}$$

  Note the WSS assumption.

- The function $\psi_g(0;\tau) \equiv \psi_g(\tau)$ is called the **multipath intensity profile** or **power delay profile**.

- The **average delay** μ_τ is the mean value of $\psi_g(\tau)$, i.e.,

$$\mu_\tau = \frac{\int_0^\infty \tau \psi_g(\tau) \mathrm{d}\tau}{\int_0^\infty \psi_g(\tau)\mathrm{d}\tau}$$

- The rms **delay spread** σ_τ is defined as the variance of $\psi_g(\tau)$, i.e.,

$$\sigma_\tau = \sqrt{\frac{\int_0^\infty (\tau - \mu_\tau)^2 \psi_g(\tau)\mathrm{d}\tau}{\int_0^\infty \psi_g(\tau)\mathrm{d}\tau}}$$

System Correlation Function

- The time autocorrelation function of the channel output $\tilde{r}(t)$ is

$$\begin{aligned}
\phi_{\tilde{r}\tilde{r}}(t, t+\Delta_t) &= \frac{1}{2}\mathrm{E}\left[\tilde{r}^*(t)\tilde{r}(t+\Delta_t)\right] \\
&= \frac{1}{2}\mathrm{E}\left[\int_0^t g^*(t,\alpha)\tilde{s}^*(t-\alpha)\mathrm{d}\alpha \times \int_0^{t+\Delta_t} g(t+\Delta_t,\beta)\tilde{s}(t+\Delta_t-\beta)\mathrm{d}\beta\right] \\
&= \int_0^t \int_0^{t+\Delta_t} \mathrm{E}\left[g^*(t,\alpha)g(t+\Delta_t,\beta)\right]\frac{1}{2}\mathrm{E}\left[\tilde{s}^*(t-\alpha)\tilde{s}(t+\Delta_t-\beta)\right]\mathrm{d}\alpha\mathrm{d}\beta \\
&= \int_0^t \int_0^{t+\Delta_t} \psi_g(\Delta_t;\alpha)\delta(\beta-\alpha)\frac{1}{2}\mathrm{E}\left[\tilde{s}^*(t-\alpha)\tilde{s}(t+\Delta_t-\beta)\right]\mathrm{d}\alpha\mathrm{d}\beta \\
&= \int_0^t \psi_g(\Delta_t;\alpha)\frac{1}{2}\mathrm{E}\left[\tilde{s}^*(t-\alpha)\tilde{s}(t+\Delta_t-\alpha)\right]\mathrm{d}\alpha \\
&= \int_0^t \psi_g(\Delta_t;\alpha)\phi_{\tilde{s}\tilde{s}}(t-\alpha, t-\alpha+\Delta_t)\mathrm{d}\alpha \\
&= \psi_g(\Delta_t;t) * \phi_{\tilde{s}\tilde{s}}(t, t+\Delta_t)
\end{aligned}$$

  where

$$\phi_{\tilde{s}\tilde{s}}(t, t+\Delta_t) = \frac{1}{2}\mathrm{E}\left[\tilde{s}^*(t)\tilde{s}(t+\Delta_t)\right] \quad .$$

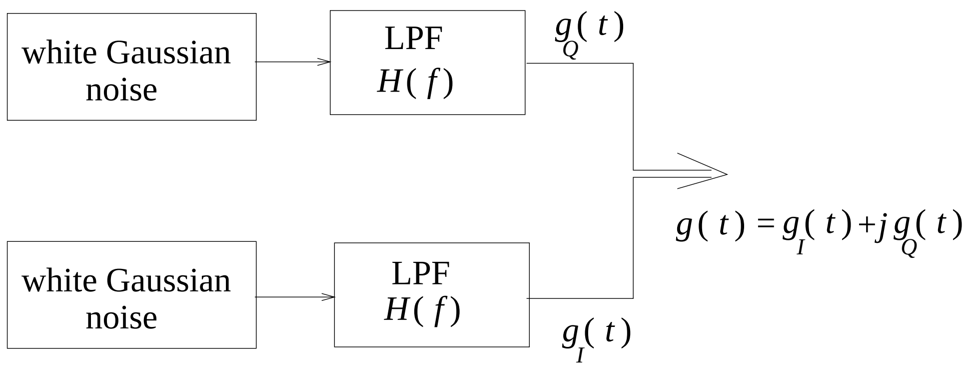
- The output time autocorrelation function is the convolution of the channel correlation function $\psi_g(\Delta_t;t)$ and the correlation function of the input waveform.

Simulation of Multipath-Fading Channels

- Computer simulation models are needed to generate the faded envelope with the statistical properties of a chosen reference model, i.e., a specified Doppler spectrum.
- Generally there are two categories of fading channel simulation models
  - Filtered-White-Noise models that pass white noise through an appropriate filter
  - Sum-of-Sinusoids models that sum together sinusoids having different amplitudes, frequencies and phases.
- Model accuracy vs. complexity is of concern
  - It is desirable to generate the faded envelope with low computational complexity while still maintaining high accuracy with respect to the chosen reference model.

Filtered White Noise

- Since the complex faded envelope can be modelled as a complex Gaussian random process, one approach for generating the complex faded envelope is to filter a white noise process with appropriately chosen low pass filters



- If the Gaussian noise sources are uncorrelated and have power spectral densities of $\Omega_p/2$ watts/Hz, and the low-pass filters have transfer function $H(f)$, then

$$
\begin{aligned}
S_{g_I g_I}(f) &= S_{g_Q g_Q}(f) = \frac{\Omega_p}{2}|H(f)|^2 \\
S_{g_I g_Q}(f) &= 0
\end{aligned}
$$

- Two approaches: IIR filtering method and IFFT filtering method

IIR Filtering Method

- implement the filters in the time domain as finite impulse response (FIR) or infinite impulse response (IIR) filters. There are two main challenges with this approach.
  - the normalized Doppler frequency, $\hat{f}_m = f_m T_s$, where T_s is the simulation step size, is very small.
    * This can be overcome with an infinite impulse response (IIR) filter designed at a lower sampling frequency followed by an interpolator to increase the sampling frequency.
  - The second main challenge is that the square-root of the target Doppler spectrum for 2-D isotropic scattering and an isotropic antenna is irrational and, therefore, none of the straightforward filter design methods can be applied.
    * One possibility is to use the MATLAB function `iirlpnorm` to design the required filter.

IIR Filtering Method

- Here we consider an IIR filter of order $2K$ that is synthesized as the the cascade of K Direct-Form II second-order (two poles and two zeroes) sections (biquads) having the form

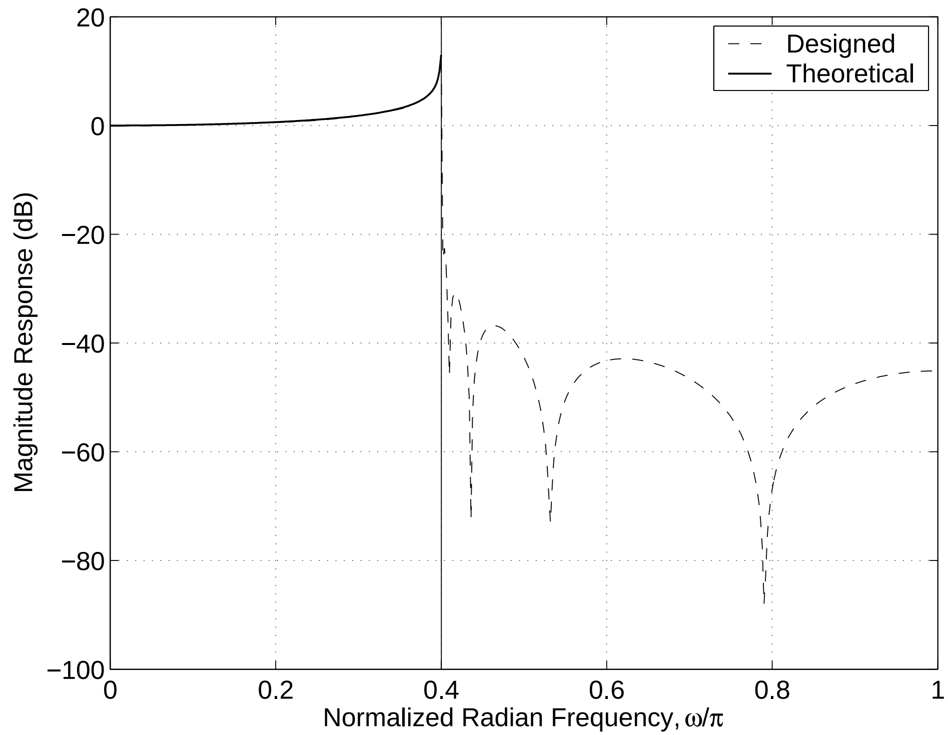
$$H(z) = A \prod_{k=1}^{K} \frac{1 + a_k z^{-1} + b_k z^{-2}}{1 + c_k z^{-1} + d_k z^{-2}} \ .$$

For example, for $f_m T_s = 0.4$, $K = 5$, and an ellipsoidal accuracy of 0.01, we obtain the coefficients tabulated below

Coefficients for $K = 5$ biquad stage elliptical filter, $f_m T_s = 0.4$, $K = 5$

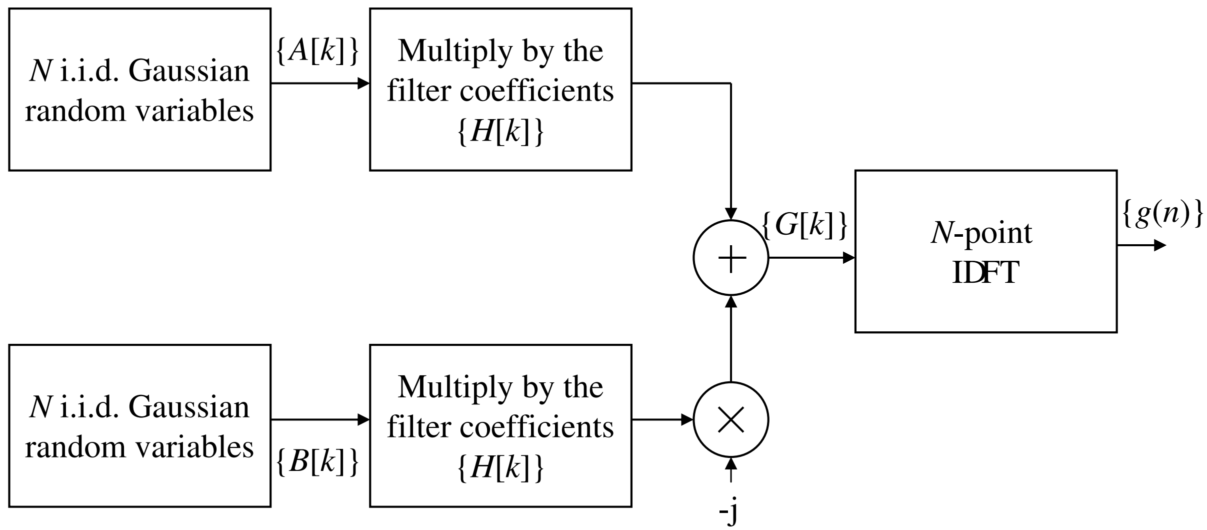
Stage	*Filter Coefficients*			
k	a_k	b_k	c_k	d_k
1	1.5806655278853	0.99720549234156	-0.64808639835819	0.88900798545419
2	0.19859624284546	0.99283177405702	-0.62521063559242	0.97280125737779
3	-0.60387555371625	0.9999939585621	-0.62031415619505	0.99996628706514
4	-0.56105447536557	0.9997677910713	-0.79222029531477	0.2514924845181
5	-0.39828788982331	0.99957862369507	-0.71405064745976	0.64701702807931
A	0.020939537466725			

IIR Filtering Method



Magnitude response of the designed shaping filter, $f_m T_s = 0.4$, $K = 5$.

IFFT Filtering Method



IDFT-based fading simulator.

- To implement 2-D isotropic scattering, the filter $H[k]$ can be specified as follows:

$$H[k] = \begin{cases} 0 & , \quad k = 0 \\ \sqrt{\frac{1}{2\pi f_m \sqrt{1-(k/(N\hat{f}_m))^2}}} & , \quad k = 1, 2, \ldots, k_m - 1 \\ \sqrt{k_m \left[\frac{\pi}{2} - \arctan\left(\frac{k_m-1}{\sqrt{2k_m-1}}\right)\right]} & , \quad k = k_m \\ 0 & , \quad k = k_m + 1, \ldots, N - k_m - 1 \\ \sqrt{k_m \left[\frac{\pi}{2} - \arctan\left(\frac{k_m-1}{\sqrt{2k_m-1}}\right)\right]} & , \quad k = N - k_m \\ \sqrt{\frac{1}{2\pi f_m \sqrt{1-(N-k/(N\hat{f}_m))^2}}} & , \quad N - k_m + 1, \ldots, N - 1 \end{cases}$$

- One problem with the IFFT method is that the faded envelope is discontinuous from one block of N samples to the next.

Sum of Sinusoids (SoS) Methods - Clarke's Model

- With N equal strength ($C_n = \sqrt{1/N}$) arriving plane waves

$$
\begin{aligned}
g(t) &= \sqrt{1/N} \sum_{n=1}^{N} \mathrm{e}^{j(2\pi f_m t \cos\theta_n + \hat{\phi}_n)} \\
&= \sqrt{1/N} \sum_{n=1}^{N} \cos(2\pi f_m t \cos\theta_n + \hat{\phi}_n) + j\sqrt{1/N} \sum_{n=1}^{N} \sin(2\pi f_m t \cos\theta_n + \hat{\phi}_n) \qquad (1) \\
&= g_I(t) + j g_Q(t)
\end{aligned}
$$

$$. \qquad (2)$$

- The normalization $C_n = \sqrt{1/N}$ makes $\Omega_p = 1$.
- The phases $\hat{\phi}_n$ are independent and uniform on $[-\pi, \pi)$.
- With 2-D isotropic scattering, the θ_n are also independent and uniform on $[-\pi, \pi)$, and are independent of the $\hat{\phi}_n$.
- Types of SoS simulators
  - deterministic - $\{\theta_n\}$ and $\{\hat{\phi}_n\}$ are fixed for all simulation runs.
  - statistical - either $\{\theta_n\}$ or $\{\hat{\phi}_n\}$, or both, are random for each simulation run.
  - ergodic statistical - either $\{\theta_n\}$ or $\{\hat{\phi}_n\}$, or both, are random, but only a single simulation run is required.

Clarke's Model - Ensemble Averages

- The statistical properties of Clarke's model in for *finite* N are

$$\begin{aligned}\phi_{g_I g_I}(\tau) &= \phi_{g_Q g_Q}(\tau) = \frac{1}{2} J_0(2\pi f_m \tau) \\ \phi_{g_I g_Q}(\tau) &= \phi_{g_Q g_I}(\tau) = 0 \\ \phi_{gg}(\tau) &= \frac{1}{2} J_0(2\pi f_m \tau) \\ \phi_{|g|^2|g|^2}(\tau) &= \mathrm{E}[|g|^2(t)|g|^2(t+\tau)] \\ &= 1 + \frac{N-1}{N} J_0^2(2\pi f_m \tau)\end{aligned}$$

- For finite N, the ensemble averaged auto- and cross-correlation of the quadrature components match those of the 2-D isotropic scattering reference model.
- The squared envelope autocorrelation reaches the desired form $1 + J_0^2(2\pi f_m \tau)$ asymptotically as $N \to \infty$.

Clarke's Model - Time Averages

- In simulations, time averaging is often used in place of ensemble averaging. The corresponding time average correlation functions $\hat{\phi}(\cdot)$ (all time averaged quantities are distinguished from the statistical averages with a '$\hat{}$') are random and depend on the specific realization of the random parameters in a given simulation trial.

- The variances of the time average correlation functions, defined as

$$\mathrm{Var}[\hat{\phi}(\cdot)] = \mathrm{E}\Big[\Big| \hat{\phi}(\cdot) \; - \; \lim_{N \to \infty} \; \phi(\cdot) \Big|^2 \Big] \; ,$$

characterizes the closeness of a simulation trial with finite N and the ideal case with $N \to \infty$.

- These variances can be derived as follows:

$$\begin{aligned}
\mathrm{Var}[\hat{\phi}_{g_I g_I}(\tau)] &= \mathrm{Var}[\hat{\phi}_{g_Q g_Q}(\tau)] \\
&= \frac{1 + J_0(4\pi f_m \tau) - 2J_0^2(2\pi f_m \tau)}{8N} \\
\mathrm{Var}[\hat{\phi}_{g_I g_Q}(\tau)] &= \mathrm{Var}[\hat{\phi}_{g_Q g_I}(\tau)] \\
&= \frac{1 - J_0(4\pi f_m \tau)}{8N} \\
\mathrm{Var}[\hat{\phi}_{gg}(\tau)] &= \frac{1 - J_0^2(2\pi f_m \tau)}{4N}
\end{aligned}$$

Jakes' Deterministic Method

- To approximate an isotropic scattering channel, it is assumed that the N arriving plane waves uniformly distributed in angle of incidence:

$$\theta_n = 2\pi n/N \ , \quad n = 1,\ 2, \ldots,\ N$$

- By choosing $N/2$ to be an odd integer, the sum in (2) can be rearranged into the form

$$g(t) = \sqrt{\frac{1}{N}} \sum_{n=1}^{N/2-1} \left[e^{-j(2\pi f_m t \cos\theta_n + \hat{\phi}_{-n})} + e^{j(2\pi f_m t \cos\theta_n + \hat{\phi}_n)} \right] + e^{-j(2\pi f_m t + \hat{\phi}_{-N})} + e^{j(2\pi f_m t + \hat{\phi}_N)} \qquad (3)$$

- The Doppler shifts progress from $-2\pi f_m \cos(2\pi/N)$ to $+2\pi f_m \cos(2\pi/N)$ as n progresses from 1 to $N/2-1$ in the first sum, while in the second sum they progress from $+2\pi f_m \cos(2\pi/N)$ to $-2\pi f_m \cos(2\pi/N)$.

- Jakes uses nonoverlapping frequencies to write $g(t)$ as

$$g(t) = \sqrt{2}\sqrt{\frac{1}{N}} \sum_{n=1}^{M} \left[e^{-j(\hat{\phi}_{-n} + 2\pi f_m t \cos\theta_n)} + e^{j(\hat{\phi}_n + 2\pi f_m t \cos\theta_n)} \right] + e^{-j(\hat{\phi}_{-N} + 2\pi f_m t)} + e^{j(\hat{\phi}_N + 2\pi f_m t)} \qquad (4)$$

where

$$M = \frac{1}{2}\left(\frac{N}{2} - 1\right)$$

and the factor $\sqrt{2}$ is included so that the total power remains unchanged.

- Note that (3) and (4) are not equal. In (3) all phases are independent. However, (4) implies that $\hat{\phi}_n = -\hat{\phi}_{-N/2+n}$ and $\hat{\phi}_{-n} = -\hat{\phi}_{N/2-n}$ for $n = 1, \ldots, M$. This introduces correlation into the phases

- Jakes' further imposes the constraint $\hat{\phi}_n = -\hat{\phi}_{-n}$ and $\hat{\phi}_N = -\hat{\phi}_{-N}$ (but with further correlation introduced in the phases) to give

$$
\begin{aligned}
g(t) \;=\; & \sqrt{\frac{2}{N}}\left\{\left[2\sum_{n=1}^{M}\cos\beta_n\cos 2\pi f_n t + \sqrt{2}\cos\alpha\cos 2\pi f_m t\right]\right. \\
& \left. + j\left[2\sum_{n=1}^{M}\sin\beta_n\cos 2\pi f_n t + \sqrt{2}\sin\alpha\cos 2\pi f_m t\right]\right\}
\end{aligned}
$$

where

$$\alpha = \hat{\phi}_N = \qquad \beta_n = \hat{\phi}_n$$

- Time averages:

$$
\begin{aligned}
< g_I^2(t) > &= \frac{2}{N}\left[2\sum_{n=1}^{M}\cos^2\beta_n + \cos^2\alpha\right] \\
&= \frac{2}{N}\left[M + \cos^2\alpha + \sum_{n=1}^{M}\cos 2\beta_n\right]
\end{aligned}
$$

$$
\begin{aligned}
< g_Q^2(t) > &= \frac{2}{N}\left[2\sum_{n=1}^{M}\sin^2\beta_n + \sin^2\alpha\right] \\
&= \frac{2}{N}\left[M + \sin^2\alpha - \sum_{n=1}^{M}\cos 2\beta_n\right]
\end{aligned}
$$

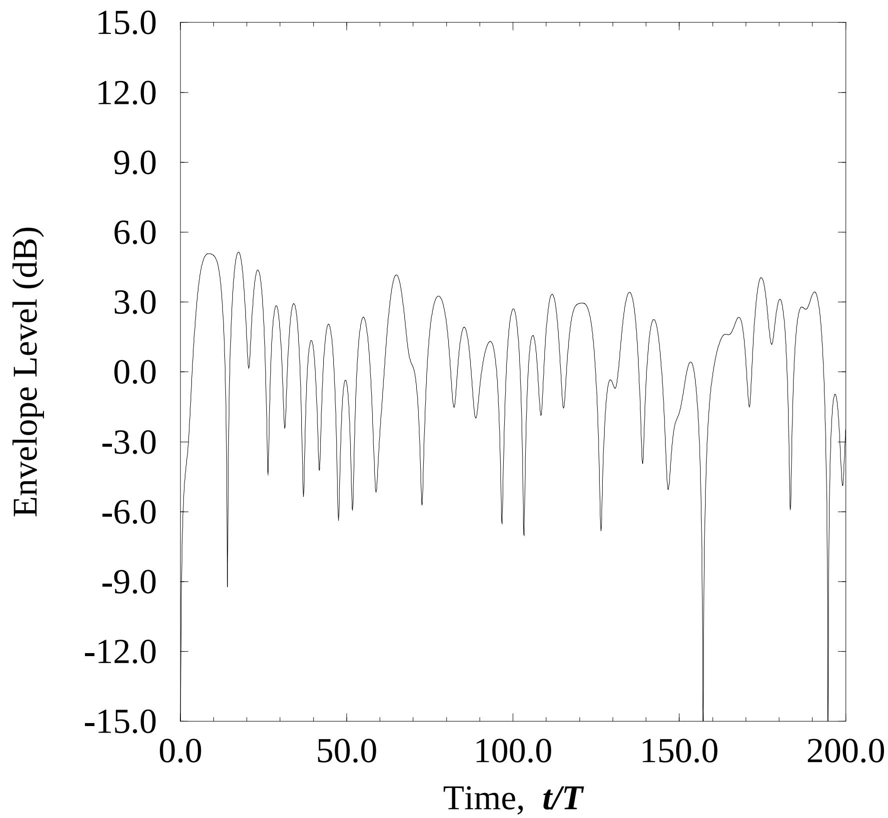
$$
< g_I(t)g_Q(t) > = \frac{2}{N}\left[2\sum_{n=1}^{M}\sin\beta_n\cos\beta_n + \sin\alpha\cos\alpha\right] \ .
$$

- Choose the β_n and α so that $g_I(t)$ and $g_Q(t)$ have zero-mean, equal variance, and zero cross-correlation.

- The choices $\alpha = 0$ and $\beta_n = \pi n/M$ will yield $< g_Q^2(t) >= M/(2M+1)$, $< g_I^2(t) >= (M+1)/(2M+1)$, and $< g_I(t)g_Q(t) >= 0$.

- Note the small imbalance in the values of $< g_Q^2(t) >$ and $< g_I^2(t) >$.

- The envelope power is $< g_I^2(t) > + < g_Q^2(t) >= \Omega_p = 1$. The envelope power can be changed to any other desired value by scaling $g(t)$, i.e., $\sqrt{\Omega_p}g(t)$ will have envelope power Ω_p.



Typical faded envelope generated with 8 oscillators and $f_m T = 0.1$, where T seconds is the simulation step size.

Auto- and Cross-correlations

- The normalized autocorrelation function is

$$\phi_{gg}^{n}(\tau) = \frac{\mathrm{E}[g^*(t)g(t+\tau)]}{\mathrm{E}[|g(t)|^2]}$$

- With 2-D isotropic scattering

$$\begin{aligned}\phi_{g_I g_I}(\tau) &= \phi_{g_Q g_Q}(\tau) = \frac{\Omega_p}{2} J_0\left(2\pi f_m \tau\right) \\ \phi_{g_I g_Q}(\tau) &= \phi_{g_Q g_I}(\tau) = 0\end{aligned}$$

- Therefore,

$$\begin{aligned}\phi_{gg}^{n}(\tau) &= \frac{\mathrm{E}[g^*(t)g(t+\tau)]}{\mathrm{E}[|g(t)|^2]} \\ &= J_0\left(2\pi f_m \tau\right)\end{aligned}$$

Auto- and Cross-correlations

- For Clarke's model with angles θ_n that are independent and uniform on $[-\pi, \pi)$, the normalized autocorrelation function is

$$\phi_{gg}^n(\tau) = \frac{\mathrm{E}[g^*(t)g(t+\tau)]}{\mathrm{E}[|g(t)|^2]} = J_0(2\pi f_m \tau) \ .$$

- Clark's model with even N and the restriction $\theta_n = \frac{2\pi n}{N}$, yields the normalized ensemble averaged autocorrelation function

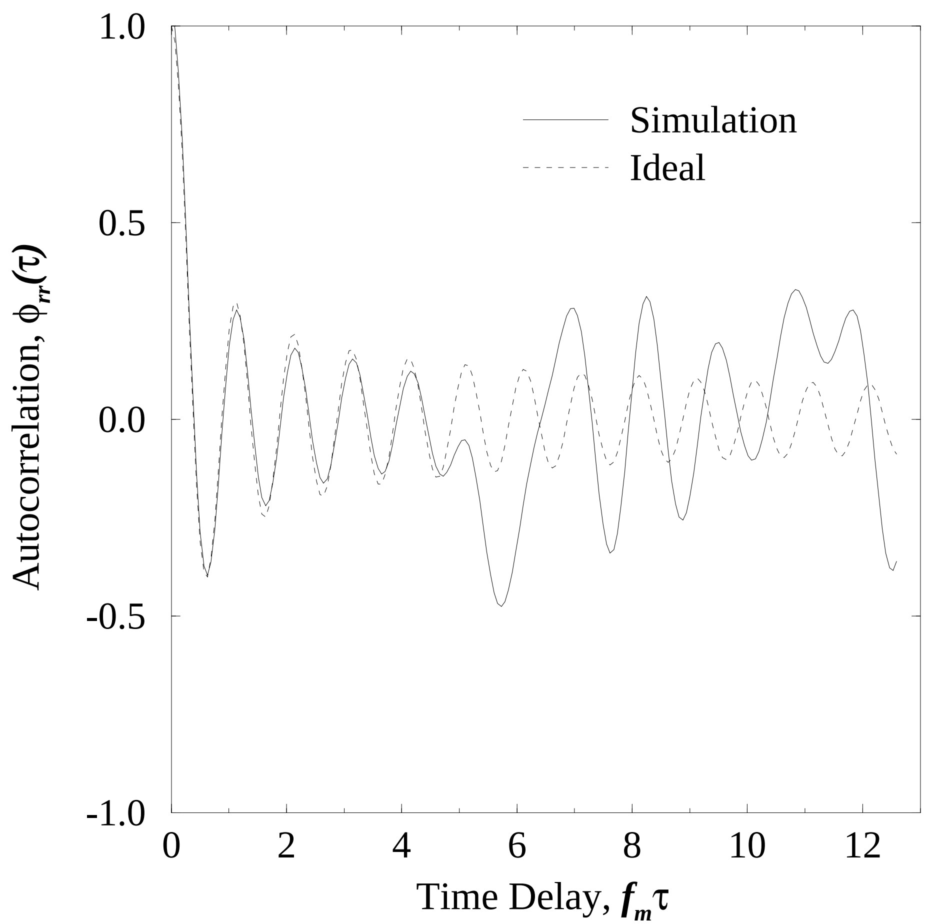
$$\phi_{gg}^n(\tau) = \frac{1}{2N} \sum_{n=1}^{N} \cos\left(2\pi f_m \tau \cos \frac{2\pi n}{N}\right) \ .$$

  – Clark's model with $\theta_n = \frac{2\pi n}{N}$ yields an autocorrelation function that deviates from the desired values at large lags.

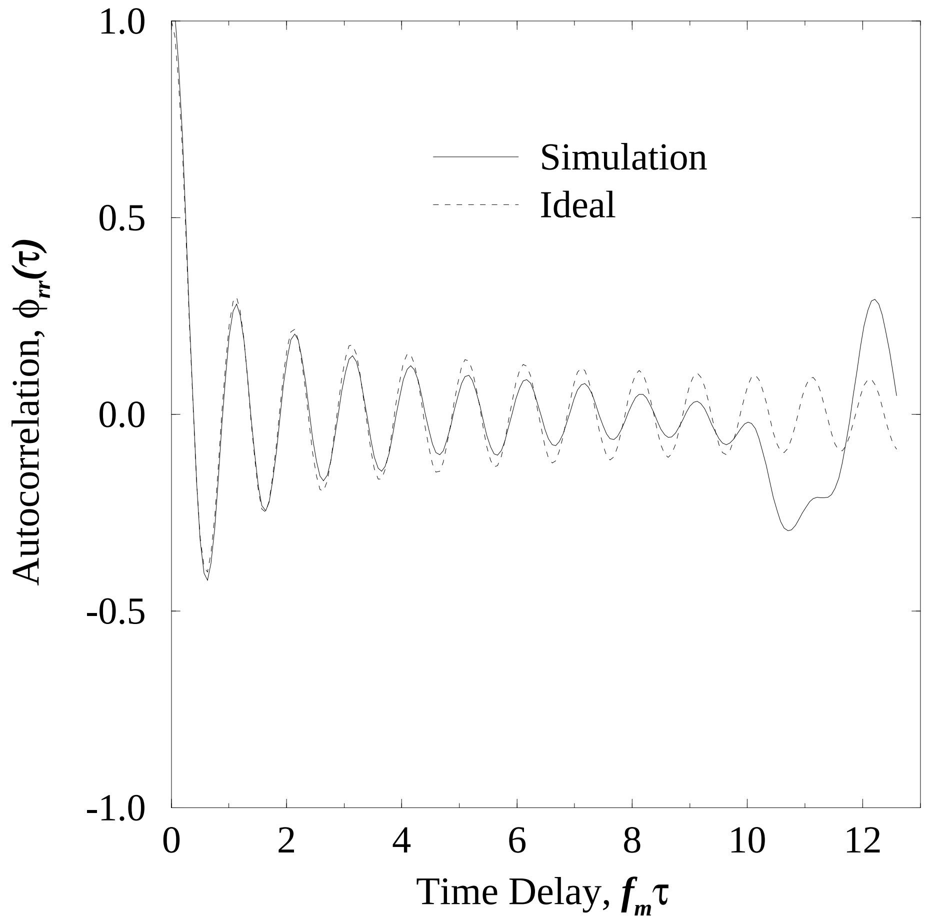
- Finally, the normalized time averaged autocorrelation function for Jakes' method is

$$\begin{aligned} \phi_{gg}^n(t, t+\tau) \ &= \ \frac{1}{2N}\left(\cos 2\pi f_m \tau + \cos 2\pi f_m(2t+\tau)\right) \\ &+ \frac{1}{N}\sum_{n=1}^{M}\left(\cos 2\pi f_n \tau + \cos 2\pi f_n(2t+\tau)\right) \end{aligned}$$

  – Jakes' fading simulator is not stationary or even wide-sense stationary.



Autocorrelation of inphase and quadrature components obtained with Clarke's method, using $\theta_n = \frac{2\pi n}{N}$ and $N = 8$ oscillators.



Autocorrelation of inphase and quadrature components obtained with Clarke's method, using $\theta_n = \frac{2\pi n}{N}$ and $N = 16$ oscillators.