EE6604
Personal & Mobile Communications

Week 12

Coherent Detection

Detection and Error Probability on Flat Fading Channels

Reading: 5.1-5.7

Optimum Coherent Detection

Consider a set of M bandpass signals with complex envelopes $\{\tilde{s}_1(t), \tilde{s}_2(t), \ldots, \tilde{s}_M(t)\}$, such that one waveform is transmitted every T seconds.

Transmit one of the M signals, say $\tilde{s}_n(t)$, over a flat fading channel with AWGN.

The received complex envelope is

$$\tilde{r}(t) = g\tilde{s}_n(t) + \tilde{n}(t)$$

where $g = \alpha e^{j\phi}$.

The AWGN $\tilde{n}(t)$ has one-sided power spectral density N_o watts/Hz and autocorrelation function function $\phi_{\tilde{n}\tilde{n}}(\tau) = N_o\delta(\tau)$.

Problem: Suppose that g is known at the receiver; channel estimation is another problem. By observing $\tilde{r}(t)$ determine which signal was transmitted.

Correlation Detector

The signal set $\{\tilde{s}_1(t), \tilde{s}_1(t), \ldots, \tilde{s}_M(t)\}$ can be expressed in terms of a complete set of orthonormal basis functions $\{\varphi_1(t), \varphi_2(t), \ldots, \varphi_N(t)\}$, where N is the dimension of the signal space.

The basis functions do not span the noise space, i.e., the noise $\tilde{n}(t)$ waveform cannot be represented exactly in terms of the basis functions. As will be shown later, the component of the noise process that falls outside of the signal space is irrelevant to the detection of the signal. It follows that

$$\begin{aligned}\tilde{r}_i &= \int_{-\infty}^{\infty} \tilde{r}(t)\varphi_i^*(t)dt \\ &= g\int_{-\infty}^{\infty} \tilde{s}_n(t)\varphi_i^*(t)dt + \int_{-\infty}^{\infty} \tilde{n}(t)\varphi_i^*(t)dt \\ &= g\tilde{s}_{n_i} + \tilde{n}_i\end{aligned}$$

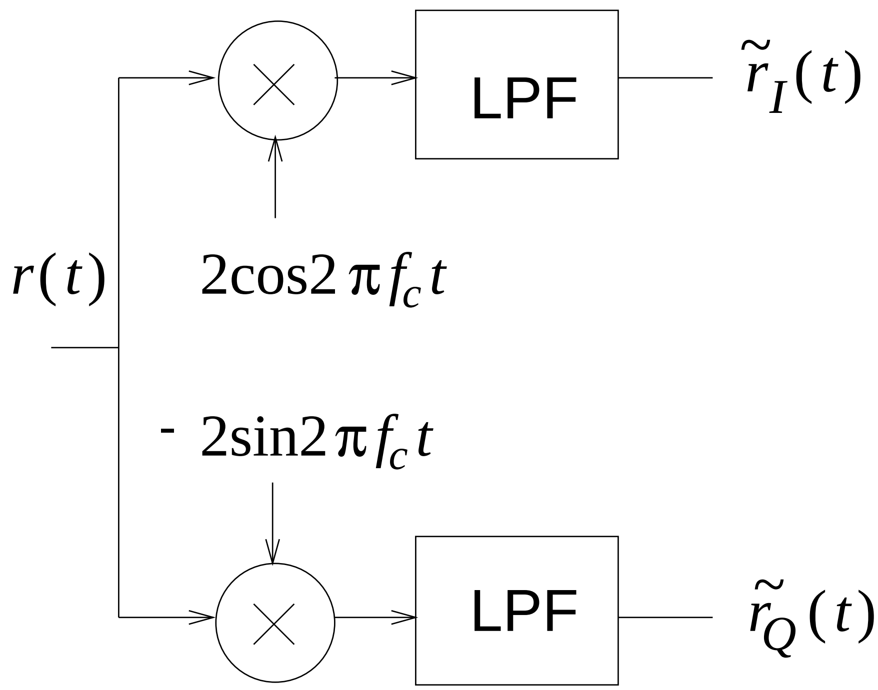
Hence, the projection of the received complex envelope $\tilde{r}(t)$ onto the signal space yields the vector

$$\tilde{\mathbf{r}} = g\tilde{\mathbf{s}}_n + \tilde{\mathbf{n}} \ ,$$

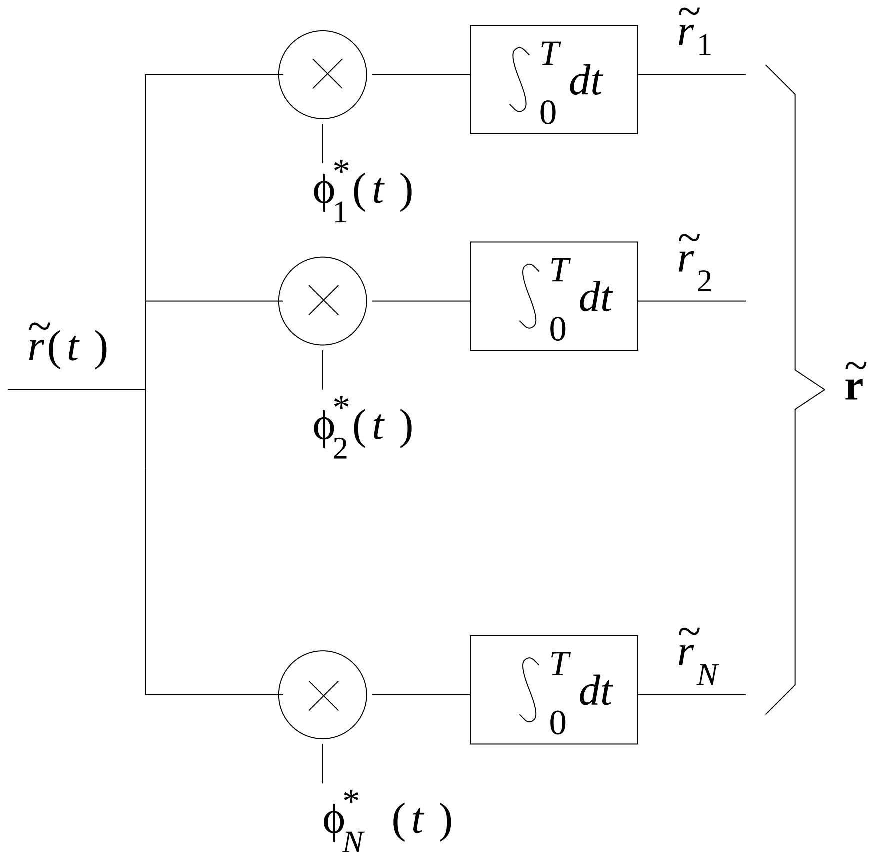
where

$$\begin{aligned}\tilde{\mathbf{r}} &= (\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_N) \\ \tilde{\mathbf{s}}_n &= (\tilde{s}_{n_1}, \tilde{s}_{n_2}, \ldots, \tilde{s}_{n_N}) \\ \tilde{\mathbf{n}} &= (\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_N) \ .\end{aligned}$$

Quadrature Demodulator



Correlation Detector



Matched Filter Receiver

Suppose that the received complex envelope $\tilde{r}(t)$ is filtered with a bank of matched filters having the impulse responses

$$h_i(t) = \varphi_i^*(T_o - t) \quad , \quad 0 \leq t \leq T_o$$

and sample the filter outputs at time $t = T_o$. Here T_o is the length of the shaping pulse.

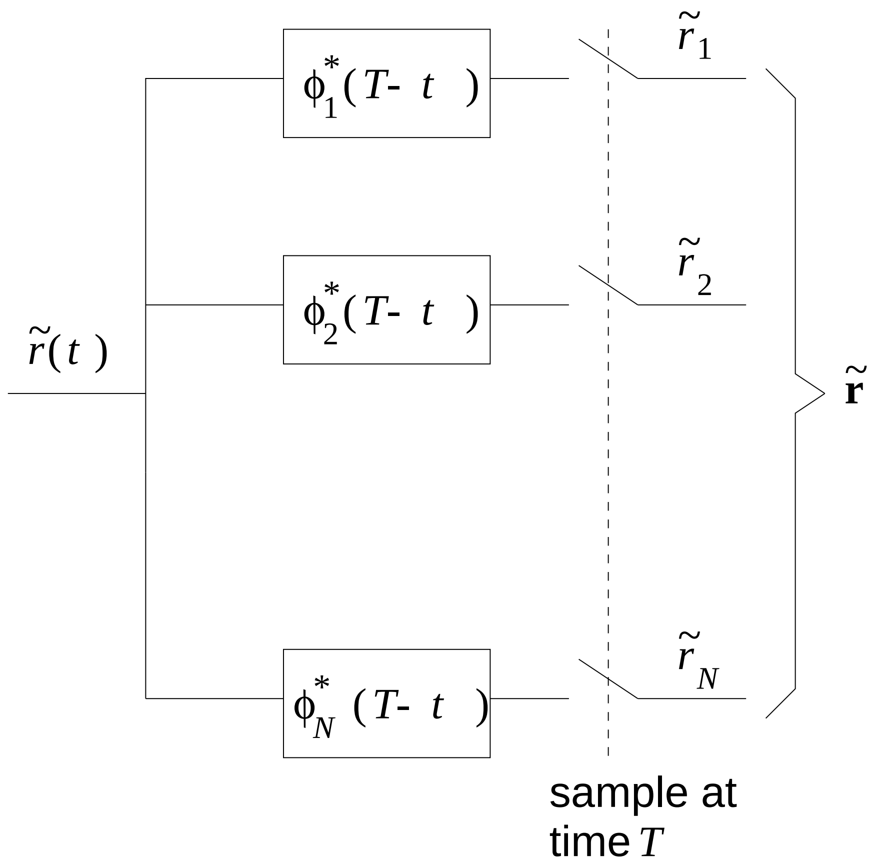
The filter outputs are

$$\begin{aligned} y_i(t) &= \int_0^t \tilde{r}(\tau) h_i(t-\tau) d\tau \\ &= \int_0^t \tilde{r}(\tau) \varphi_i^*(T_o - t + \tau) d\tau \end{aligned}$$

$$y_i = y_i(T_o) = \int_0^{T_o} \tilde{r}(\tau) \varphi_i^*(\tau) d\tau$$

Note that $y_i = \tilde{r}_i$, i.e., the matched filter outputs are identical to the correlation detector outputs.

Matched Filter Receiver



Noise Statistics

The noise components $\tilde{n}_k$ have mean

$$\mathrm{E}[\tilde{n}_k] = \int_{-\infty}^{\infty} \mathrm{E}[\tilde{n}(t)]\varphi_k^*(t)dt = 0$$

and covariance

$$\begin{aligned} \mu_{jk} &= \frac{1}{2}\mathrm{E}[\tilde{n}_j\tilde{n}_k^*] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2}\mathrm{E}[\tilde{n}(t)\tilde{n}^*(s)]\varphi_j^*(t)\varphi_k(s)dtds \\ &= N_o \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(t-s)\varphi_j^*(t)\varphi_k(s)dtds \\ &= N_o \int_{-\infty}^{\infty} \varphi_j^*(t)\varphi_k(t)dt \\ &= N_o\delta_{jk} \end{aligned}$$

Therefore, the $\tilde{n}_k$ are independent zero mean complex Gaussian random variables with variance N_o.

Joint Conditional Density

- The vector $\tilde{\mathbf{n}}$ has the joint multivariate complex Gaussian density function

$$\begin{aligned} p(\tilde{\mathbf{n}}) &= \prod_{i=1}^{N} \frac{1}{2\pi N_o} \exp\left\{ -\frac{1}{2N_o} |\tilde{n}_i|^2 \right\} \\ &= \frac{1}{(2\pi N_o)^N} \exp\left\{ -\frac{1}{2N_o} \|\tilde{\mathbf{n}}\|^2 \right\} \quad . \end{aligned}$$

- Hence, the vector $\tilde{\mathbf{r}}$ has the joint conditional density function

$$p(\tilde{\mathbf{r}}|g, \tilde{\mathbf{s}}_m) = \frac{1}{(2\pi N_o)^N} \exp\left\{ -\frac{1}{2N_o} \|\tilde{\mathbf{r}} - g\tilde{\mathbf{s}}_m\|^2 \right\}$$

which is a multivariate complex Gaussian distribution with mean $g\tilde{\mathbf{s}}_m$.

Noise Remainder Process

- Note that

$$\tilde{r}(t) = \sum_{k=0}^{N-1} \tilde{r}_k \varphi_k(t) + \tilde{z}(t)$$

  where $\tilde{z}(t)$ is the **remainder process**

$$\tilde{z}(t) = \tilde{n}(t) - \sum_{k=0}^{N-1} \tilde{n}_k \varphi_k(t)$$

- The remainder process is outside the vector space spanned by the basis functions $\{\varphi_k\}$.
- Is $\tilde{z}(t)$ important for the detection problem?

Irrelevance

Is the vector $\tilde{\mathbf{r}}$ uncorrelated with $\tilde{z}(t)$ and, hence, irrelevant? It follows that

$$\begin{aligned}
\frac{1}{2}\mathrm{E}[\tilde{z}(t)r_j^*] &= \frac{1}{2}\mathrm{E}[\tilde{z}(t)]g\tilde{s}_{m_j}^* + \frac{1}{2}\mathrm{E}[\tilde{z}(t)\tilde{n}_j^*] \\
&= \frac{1}{2}\mathrm{E}[\tilde{z}(t)\tilde{n}_j^*] \\
&= \frac{1}{2}\mathrm{E}\left[\left(\tilde{n}(t) - \sum_{n=0}^{N-1}\tilde{n}_n\varphi_n(t)\right)\tilde{n}_j^*\right] \\
&= \int_{-\infty}^{\infty}\frac{1}{2}\mathrm{E}[\tilde{n}(t)\tilde{n}^*(\tau)]\varphi_j(\tau)d\tau - \sum_{n=0}^{N-1}\frac{1}{2}\mathrm{E}[\tilde{n}_n\tilde{n}_j^*]\varphi_n(t) \\
&= N_o\varphi_j(t) - N_o\varphi_j(t) = 0
\end{aligned}$$

Hence, the vector $\tilde{\mathbf{r}}$ is uncorrelated with $\tilde{z}(t)$ and, therefore, $\tilde{z}(t)$ is irrelevant since it does not contain any information about $\tilde{\mathbf{r}}$.

This is Wozencraft's irrelevance theorem which is certainly not irrelevant!

Optimum Decision Rule

Suppose that $\tilde{s}_m(t)$ is transmitted and the correlation or matched filter receiver outputs the vector $\tilde{\mathbf{r}} = g\tilde{\mathbf{s}}_m + \tilde{\mathbf{n}}$.

The maximum *a posteriori* (MAP) receiver observes $\tilde{\mathbf{r}}$ and decides in favour of the message $\tilde{\mathbf{s}}_k$ that maximizes the *a posteriori* probability $P(\tilde{\mathbf{s}}_k \text{ sent}|g, \tilde{\mathbf{r}})$.

If $\tilde{\mathbf{r}}$ is received and the decision is made that $\tilde{\mathbf{s}}_k$ was sent, then the conditional probability of decision error is

$$P_{e|\tilde{\mathbf{r}}} = P(\tilde{\mathbf{s}}_k \text{ not sent}|g, \tilde{\mathbf{r}}) = 1 - P(\tilde{\mathbf{s}}_k \text{ sent}|g, \tilde{\mathbf{r}})$$

and the unconditional probability of error is

$$P_e = \int_{-\infty}^{\infty} (1 - P(\tilde{\mathbf{s}}_k \text{ sent}|g, \tilde{\mathbf{r}})\, p(g, \tilde{\mathbf{r}}) dg d\tilde{\mathbf{r}}$$

Since MAP receiver maximizes $P(\tilde{\mathbf{s}}_k \text{ sent}|g, \tilde{\mathbf{r}})$ for any and all g and $\tilde{\mathbf{r}}$ it minimizes the probability of error.

Bayes' Rule Applied

Using Bayes' rule, the *a posteriori* probability $P(\tilde{\mathbf{s}}_m|g,\tilde{\mathbf{r}})$ can be written in the form

$$P(\tilde{\mathbf{s}}_m|g,\tilde{\mathbf{r}}) = \frac{p(\tilde{\mathbf{r}}|g,\tilde{\mathbf{s}}_m)P_m}{p(\tilde{\mathbf{r}})} \quad m = 1,\ \cdots,\ M$$

Therefore, the MAP decision rule is

$$\text{choose } \tilde{\mathbf{s}}_m \text{ if } \quad p(\tilde{\mathbf{r}}|g,\tilde{\mathbf{s}}_m)P_m \geq p(\tilde{\mathbf{r}}|g,\tilde{\mathbf{s}}_{\hat{m}})P_{\hat{m}} \quad \forall\ \hat{m} \neq m$$

A receiver that chooses $\mathbf{s}_m$ to maximize $p(\tilde{\mathbf{r}}|g\tilde{\mathbf{s}}_m)$ regardless of the *a priori* message probabilities P_m is called a *maximum likelihood (ML) receiver.*

The ML decision rule is

$$\text{choose } \mathbf{s}_m \text{ if } \quad p(\tilde{\mathbf{r}}|g,\tilde{\mathbf{s}}_m) \geq p(\tilde{\mathbf{r}}|g,\tilde{\mathbf{s}}_{\hat{m}}) \quad \forall\ \hat{m} \neq m$$

If the messages are equally likely, i.e., $\mathrm{P}_m = 1/M$, then the $\mathbf{s}_m$ that maximizes $p(\tilde{\mathbf{r}}|g,\tilde{\mathbf{s}}_m)$ also maximizes $P(\tilde{\mathbf{s}}_m|g,\tilde{\mathbf{r}})$. Under this condition, the ML receiver also minimizes the probability of error.

Slow Flat Fading Channels with AWGN

- The joint conditional pdf $p(\tilde{\mathbf{r}}|g, \tilde{\mathbf{s}}_m)$ is

$$p(\tilde{\mathbf{r}}|g, \tilde{\mathbf{s}}_m) = \frac{1}{(2\pi N_o)^N} \exp\left\{-\frac{1}{2N_o}\|\tilde{\mathbf{r}} - g\tilde{\mathbf{s}}_m\|^2\right\} \quad .$$

- When the additive impairment is AWGN, the signal vector $\tilde{\mathbf{s}}_m$ that maximizes $p(\tilde{\mathbf{r}}|g, \tilde{\mathbf{s}}_m)$ will also minimizes the Euclidean distance

$$\mu_1(\tilde{\mathbf{s}}_m) = \|\tilde{\mathbf{r}} - g\tilde{\mathbf{s}}_m\|^2$$

- In other words, the ML receiver decides in favor of the scaled message vector $g\tilde{\mathbf{s}}_m$ that is closest in squared Euclidean distance to the received vector $\tilde{\mathbf{r}}$. Such a receiver is said to make **minimum distance** decisions.
- An alternative form of the ML receiver can be derived as

$$\mu_1(\tilde{\mathbf{s}}_m) = \|\tilde{\mathbf{r}}\|^2 - 2\mathrm{Re}\left\{(\tilde{\mathbf{r}}, g\tilde{\mathbf{s}}_m)\right\} + |g|^2\|\tilde{\mathbf{s}}_m\|^2$$

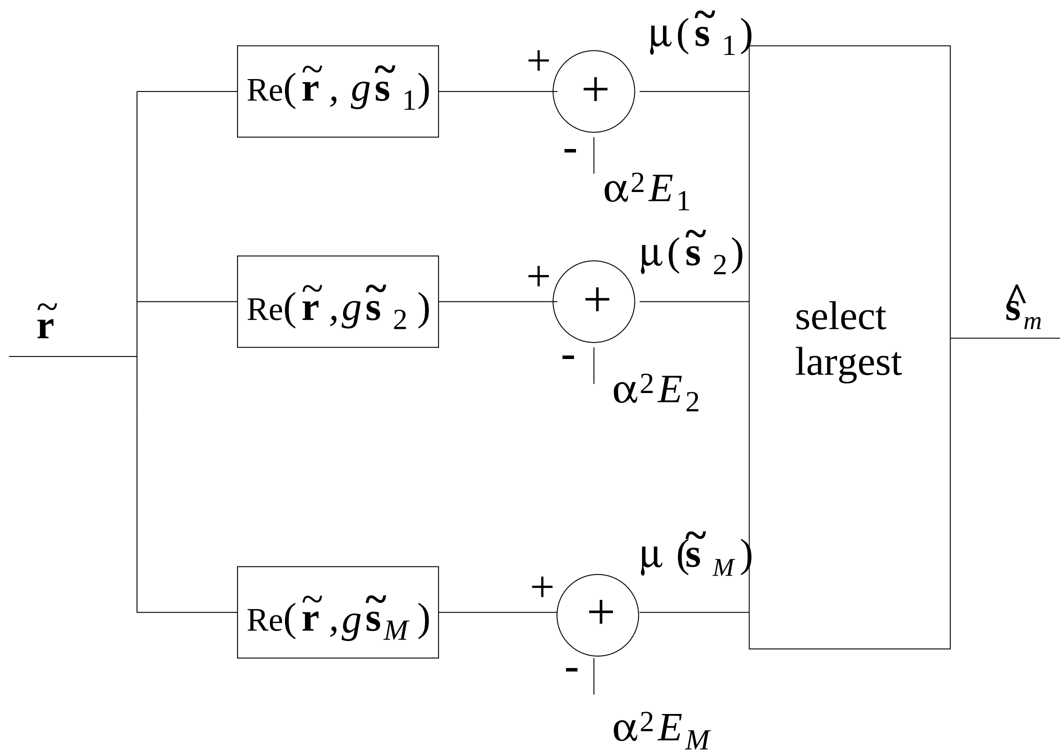
- Then notice that $\|\tilde{\mathbf{r}}\|^2$ is independent of $\tilde{\mathbf{s}}_m$ and $\|\tilde{\mathbf{s}}_m\|^2 = 2E_m$. Hence, the ML just needs to maximize the metric

$$\mu_2(\tilde{\mathbf{s}}_m) = \mathrm{Re}\left\{(\tilde{\mathbf{r}}, g\tilde{\mathbf{s}}_m)\right\} - |g|^2 E_m \quad .$$

- Using the definition of the inner product, the above decision metric can be rewritten in the alternate form

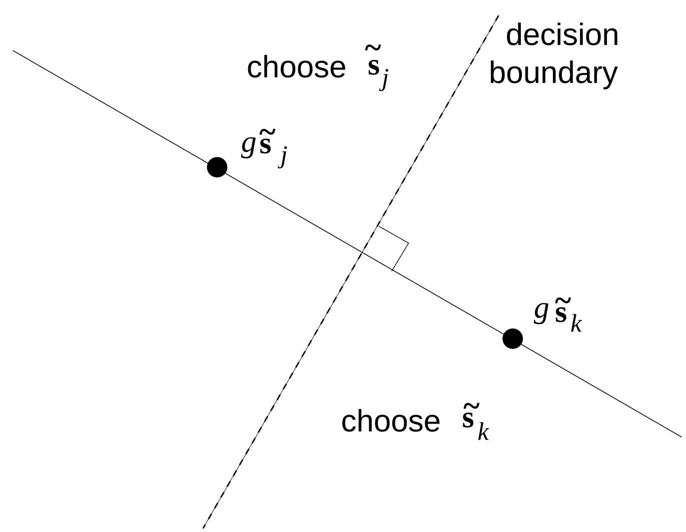
$$\begin{aligned} \mu_2(\tilde{\mathbf{s}}_m) &= \mathrm{Re}\left\{\int_{-\infty}^{\infty} \tilde{r}(t) g^* \tilde{s}_m^*(t) dt\right\} - |g|^2 E_m \\ &\equiv \mathrm{Re}\left\{\int_{-\infty}^{\infty} \tilde{r}(t) \mathrm{e}^{-j\phi} \tilde{s}_m^*(t) dt\right\} - \alpha E_m \quad , m = 1, \ldots, M \quad . \end{aligned}$$

Metric Computer for a Slow Flat Fading Channel



Pairwise Error Probability

- Consider two equally likely signal vectors $\tilde{\mathbf{s}}_j$ and $\tilde{\mathbf{s}}_k$ in a signal constellation of size M.
- The two signal vectors $\tilde{\mathbf{s}}_j$ and $\tilde{\mathbf{s}}_k$ are separated *at the receiver* by the squared Euclidean distance $\|g\tilde{\mathbf{s}}_j - g\tilde{\mathbf{s}}_k\|^2 = \alpha^2\|\tilde{\mathbf{s}}_j - \tilde{\mathbf{s}}_k\|^2$.
- A decision boundary can be established at the midpoint between the two signal vectors as shown below.



Pairwise Error Probability

- Suppose that vector $\tilde{\mathbf{s}}_j$ is sent, and let $\mathrm{P}[e|\tilde{\mathbf{s}}_j]$ denote the probability of ML decision error. This error probability is just the probability that the noise along the vector $g\tilde{\mathbf{s}}_j - g\tilde{\mathbf{s}}_k$ forces the received vector $\tilde{\mathbf{r}} = g\tilde{\mathbf{s}}_j + \tilde{\mathbf{n}}$ to cross the decision boundary.
- Due to the circularly symmetric property of the AWGN noise, the pdf of the noise vector $\tilde{\mathbf{n}}$ is invariant to its rotation about the origin in the signal space. Hence, the noise component along the line that passes through the two signal vectors will have zero mean and variance N_o.
- It follows that the error probability is equal to

$$\mathrm{P}[e|\tilde{\mathbf{s}}_j] = Q\left(\sqrt{\frac{\alpha^2 \tilde{d}_{jk}^2}{4N_o}}\right) ,$$

where $\tilde{d}_{jk}^2 = \|\tilde{\mathbf{s}}_j - \tilde{\mathbf{s}}_k\|^2$ is the squared Euclidean distance between $\tilde{\mathbf{s}}_j$ and $\tilde{\mathbf{s}}_k$.

Error Probability for BPSK, MSK, QPSK

- The error probability is

$$P_b(\gamma_b) = Q(\sqrt{2\gamma_b})$$

where $\gamma_b = \alpha^2 E/N_o$ is the received *bit energy-to-noise ratio*

- For Rayleigh fading, α is Rayleigh distributed, so that

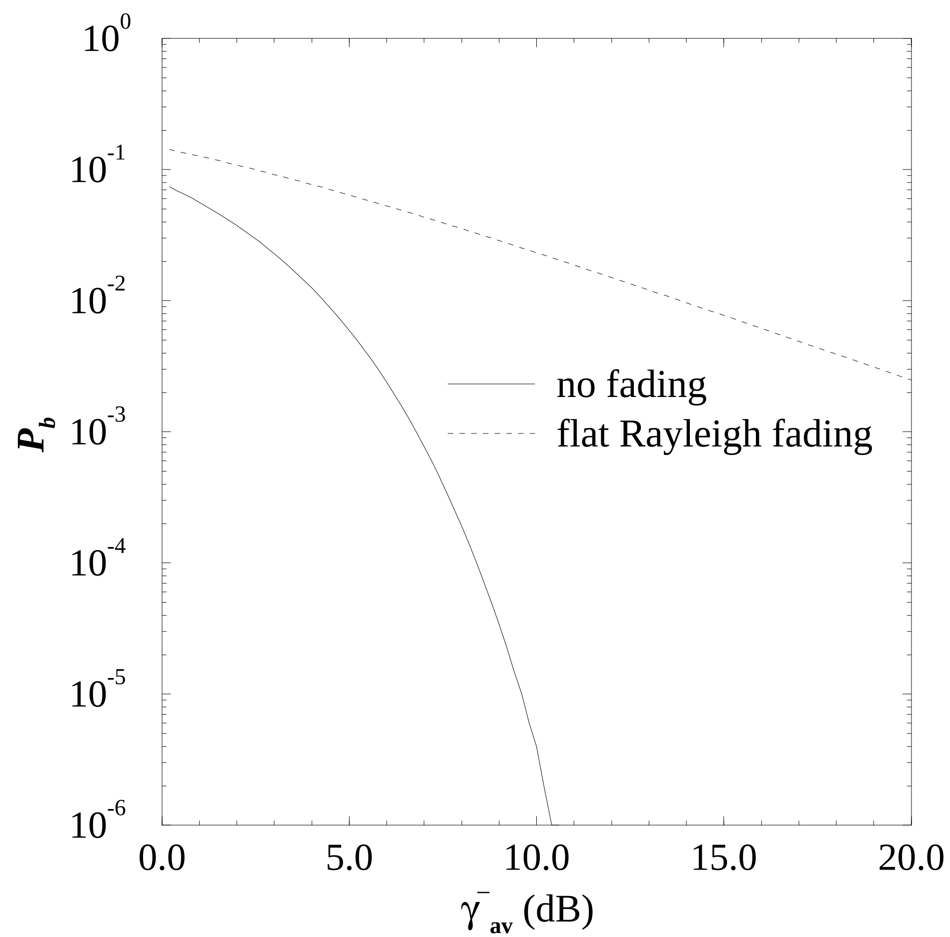
$$p_{\gamma_b}(x) = \frac{1}{\bar{\gamma}_b} e^{-x/\bar{\gamma}_b} \quad , \quad \bar{\gamma}_b = \mathrm{E}[\alpha^2]\frac{E_b}{N_o}$$

- Therefore, the average bit error probability is

$$\begin{aligned} P_b &= \int_0^\infty Q(\sqrt{2\gamma_b})p(\gamma_b)d\gamma_b \\ &= \frac{1}{2}\left[1 - \sqrt{\frac{\bar{\gamma}_b}{1+\bar{\gamma}_b}}\right] \approx \frac{1}{4\bar{\gamma}_b} \end{aligned}$$

- Fact: BPSK, MSK,and QPSK all have the same power performance.

Bit error Probability for Coherent BPSK, MSK,and QPSK



Error Probability for QAM

- Consider an M-QAM system with $M = 4^m$ for some integer m.
- The probability of symbol error is

$$P_M = 1 - (1 - P_{\sqrt{M}})^2$$

where

$$P_{\sqrt{M}} = 2\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3}{M-1}\,\gamma_{\rm s}}\right)$$

and

$$\gamma_{\rm s} = \frac{E_{\rm av}}{N_o} = \log_2 M\ \frac{E_{b\ \rm av}}{N_o}$$

is the average *received* symbol energy-to-noise ratio.

- If the channel is Rayleigh faded, the average symbol error probability is

$$P_M = \int_0^\infty P_M(x) p_{\gamma_{\rm s}}(x) dx$$

- With Gray coding the bit error probability is approximately

$$P_b \approx P_M / \log_2 M$$

Bit Error Probability for QAM