ECE 6416 Extra Problems 1

1. An average responding voltmeter is calibrated to read the correct rms voltage for a sine wave input signal. That is, the meter is calibrated to read $V_M = \left(\pi/2\sqrt{2}\right) \times \langle |v_i(t)| \rangle$.

   (a) For a square wave input with peak values $+V_P$ and $-V_P$, show that $\langle |v_i(t)| \rangle = V_P$, $\left\langle v_i^2(t) \right\rangle = V_P$, and the meter reading must be multiplied by 0.90.

   (b) For a triangle wave input with peak values $+V_P$ and $-V_P$, show that $\langle |v_i(t)| \rangle = V_P/2$, $\left\langle v_i^2(t) \right\rangle = V_P/\sqrt{3}$, and the meter reading must be multiplied by 1.04.

2. For this problem, some useful trigonometric identities are $2\sin x \cos x = \sin 2x$, $\cos(x+y) = \cos x \cos y - \sin x \sin y$, and $2\cos x \cos y = \cos(x+y) + \cos(x-y)$.

   (a) For $v_a(t) = V_A \cos\omega t$ and $v_b(t) = V_B \sin\omega t$, show that $\rho = 0$.

   (b) For $v_a(t) = V_A \cos\omega t$ and $v_b(t) = V_B \cos\omega t$, show that $\rho = 1$.

   (c) For $v_a(t) = V_A \cos\omega t$ and $v_b(t) = V_B \cos(\omega t + \varphi)$, show that $\rho = \cos\varphi$.

   (d) For $v_a(t) = V_A \cos\omega_1 t$ and $v_b(t) = V_B \cos\omega_2 t$, where $\omega_1 \neq \omega_2$, show that $\rho = 0$.

3. Make use of the results of problem 2 to show that $\left\langle [v_a(t) + v_b(t)]^2 \right\rangle$ for the four parts of problem 2 are:

   (a) $\left(V_A^2 + V_B^2\right)/2$

   (b) $(V_A + V_B)^2/2$

   (c) $\left(V_A^2 + 2V_A V_B \cos\varphi + V_B^2\right)/2$

   (d) $\left(V_A^2 + V_B^2\right)/2)$

4. An inductor L is in parallel with a series resistor R and capacitor C.

   (a) Solve for the real part of the impedance to show that the mean-square thermal noise voltage across the inductor in the frequency band Δf is given by

   $$\overline{v_n^2} = 4kTR\Delta f \frac{\left(\omega^2 LC\right)^2}{\left(1-\omega^2 LC\right)^2 + (\omega RC)^2}$$

   (b) Replace the resistor with its Thévenin noise model. Use voltage division to solve for the phasor voltage V_n across the inductor. show that $\overline{V_n V_n^*}$ gives the same answer as the one obtained above.

5. A band-pass filter has the transfer function

   $$T(s) = K \frac{(1/Q)(s/\omega_0)}{(s/\omega_0)^2 + (1/Q)(s/\omega_0) + 1}$$

   Show that the noise bandwidth of the filter is given by the integral

   $$B_n = f_0 \int_0^\infty \left[\frac{(x/Q)^2}{\left(1-x^2\right)^2 + (x/Q)^2} \right] dx$$

   where $f_0 = \omega_0/2\pi$.

6. A RC high-pass filter with $R_1 = 16\,\text{k}\Omega$ and $C_1 = 0.1\,\mu\text{F}$ and a RC low-pass filter with $R_2 = 1.6\,\text{k}\Omega$ and $C_2 = 0.01\,\mu\text{F}$ are connected in cascade with a unity gain op-amp buffer between the two to form a band-pass filter.

   (a) Show that the transfer function can be put into the form

   $$\frac{V_o}{V_i} = K\frac{(1/Q)\,(s/\omega_0)}{(s/\omega_0)^2 + (1/Q)\,(s/\omega_0) + 1}$$

   and give the equations for K, ω_0, and Q in terms of R_1, R_2, C_1, and C_2.

   (b) Show that the pole frequencies are $f_1 = 99.47\,\text{Hz}$ and $f_2 = 9.947\,\text{kHz}$.

   (c) Show that the maximum gain is $A_0 = 0.9901$.

   (d) Show that the center (geometric mean) frequency is $f_0 = 994.7\,\text{Hz}$.

   (e) Show that the quality factor is $Q = 0.09901$.

   (f) Show that the -3 dB bandwidth is $B_3 = 10.05\,\text{kHz}$.

   (g) Show that the -3 dB frequencies are $f_a = 97.54\,\text{Hz}$ and $f_b = 10.14\,\text{kHz}$.

   (h) Show that the noise bandwidth is $B_n = 15.78\,\text{kHz}$.

7. The buffer is removed from the band pass filter in problem 6. Let the voltage at the common node between the two filter sections be V_a. Write the transfer function of the filter as the product

   $$\frac{V_o}{V_i} = \frac{V_a}{V_i} \times \frac{V_o}{V_a}$$

   (a) Use voltage division to solve for both V_a/V_i (this is the hard one of the two) and V_o/V_a to show that the gain is given by

   $$\frac{V_o}{V_i} = \frac{R_1C_1s}{R_1R_2C_1C_2s^2 + [R_1C_1 + (R_1 + R_2)\,C_2]\,s + 1}$$

   (b) Put the transfer function into the form

   $$\frac{V_o}{V_i} = K\frac{(1/Q)\,(s/\omega_0)}{(s/\omega_0)^2 + (1/Q)\,(s/\omega_0) + 1}$$

   and give the equations for K, ω_0, and Q in terms of R_1, R_2, C_1, and C_2.

   (c) Show that the pole frequencies are $f_1 = 90.35\,\text{Hz}$ and $f_2 = 10.95\,\text{kHz}$.

   (d) Show that the maximum gain is $A_0 = 0.9009$.

   (e) Show that the center frequency is $f_0 = 994.7\,\text{Hz}$.

   (f) Show that the quality factor is $Q = 0.09009$.

   (g) Show that the -3 dB bandwidth is $B_3 = 11.04\,\text{kHz}$.

   (h) Show that the -3 dB frequencies are $f_a = 88.90\,\text{Hz}$ and $f_b = 11.13\,\text{kHz}$.

   (i) Show that the noise bandwidth is $B_n = 17.34\,\text{kHz}$.

8. A resistor R and an ideal capacitor C are connected in parallel. The two are in thermal equilibrium. This means that the thermal noise power generated by the resistor that is absorbed by the capacitor must equal the thermal noise power generated by the capacitor that is absorbed by the resistor. Otherwise, one would be heating up while the other is cooling off and the two would not be in thermal equilibrium.

(a) Use the Thévenin noise model of the resistor and denote its thermal phasor noise voltage by V_t, where $\overline{V_t V_t^*}$ is the mean-square noise voltage in the band Δf at the frequency of analysis. Show that the phasor noise voltage V_n across the capacitor and phasor noise voltage I_n through the capacitor are given by

$$V_n = \frac{V_t}{1 + j2\pi fRC} \qquad I_n = \frac{j2\pi fCV_t}{1 + j2\pi fRC}$$

(b) The power absorbed by the capacitor is given by $P_C = \operatorname{Re}\left(\overline{V_n I_n^*}\right)$. Show that this is zero. (Note, we are using the convention that the magnitude of a noise phasor is the rms value, not the peak value, so that there is no factor of 1/2 in the expression for P_C.)

(c) Because $P_C = 0$, it follows that the capacitor cannot absorb power from the resistor. How does this imply that the capacitor cannot generate noise power?

(d) Repeat the problem for an ideal inductor L in parallel with a resistor R.

9. A resistor R and a capacitor C are connected in parallel to form a two-terminal network.

(a) Use the Norton noise model of the resistor to solve for the phasor open-circuit noise voltage $V_{n(oc)}$ and the phasor short-circuit noise current $I_{n(sc)}$. Show that the complex correlation coefficient between $V_{n(oc)}$ and $I_{n(sc)}$ is given by

$$\gamma = \frac{\sqrt{1 + (2\pi fRC)^2}}{1 + j2\pi fRC}$$

(b) Show that the correlation impedance is

$$Z_\gamma = \frac{R}{1 + j2\pi fRC}$$

(c) *Without* using the relation $Z_\gamma Y_\gamma = |\gamma|^2$, show that the correlation admittance is

$$Y_\gamma = \frac{1 + (2\pi fRC)^2}{R(1 - j2\pi fRC)}$$

(d) Verify that $Z_\gamma Y_\gamma = |\gamma|^2$.

10. An inductor L has a series winding resistance R which can be modeled as an ideal inductor in series with a discrete resistor.

(a) Draw the circuit with a thermal phasor noise voltage V_t in series with the resistor. Calculate the short-circuit noise current $I_n = V_t/Z$, where Z is the complex impedance of the series circuit. Show that the mean-square short-circuit noise current in the band Δf is given by

$$\overline{i_n^2} = \overline{I_n I_n^*} = \frac{4kT\Delta f/R}{1 + (2\pi fL/R)^2}$$

(b) Show that the answer above can be obtained from the expression

$$\overline{i_n^2} = 4kT\operatorname{Re}(Y)\Delta f$$

where Y is the complex admittance of the circuit. Note, this result can be thought of as the dual of the formula $\overline{v_n^2} = 4kT\operatorname{Re}(Z)\Delta f$ that we used in class to calculate the mean-square open-circuit voltage of a two-terminal network.

(c) Integrate the mean-square noise current to show that the total mean-square thermal noise current generated by the inductor, i.e. the noise in the band $0 \le f \le \infty$, is

$$\overline{i_{total}^2} = \frac{kT}{L}$$

Hint: $\int dx/\left(1+x^2\right) = \tan^{-1} x$

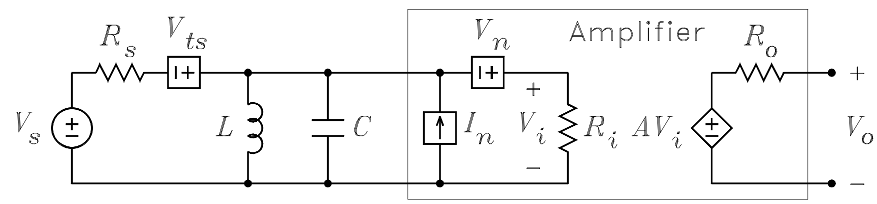
11. Given resistors R_1 and R_2, denote the thermal noise voltages generated by v_{t1} and v_{t2}, respectively. The noise voltages are independent.

(a) The resistors are connected in series. Show that the mean-square noise voltage across the series combination is the same as the mean-square thermal noise generated by a resistor of value $R_1 + R_2$.

(b) The resistors are connected in parallel. Show that the mean-square thermal noise voltage across the parallel combination is the same as the mean-square thermal noise voltage generated by a resistor of value $R_1 \| R_2$.

(c) The above two results are not sufficient to prove that the thermal noise voltage generated by any two terminal network of resistors can be obtained by reducing the network with series and parallel combinations. Give an example circuit that illustrates this.

12. An amplifier model is shown in the figure. It is given that $R_s = 1\,\mathrm{k\Omega}$, $L = 1\,\mathrm{mH}$, $C = 1\,\mu\mathrm{F}$, $R_i = 10\,\mathrm{k\Omega}$, $A = 100$, $R_o = 100\,\Omega$, $\sqrt{\overline{v_n^2}/\Delta f} = 10\,\mathrm{nV}/\sqrt{\mathrm{Hz}}$, $\sqrt{\overline{i_n^2}/\Delta f} = 1\,\mathrm{nA}/\sqrt{\mathrm{Hz}}$, and $\gamma = 0$.



(a) Show that $\sqrt{\overline{v_{ni}^2}/\Delta f} = 1.11\,\mu\mathrm{V}$ at $f = 10\,\mathrm{kHz}$.

(b) Show that $\sqrt{\overline{v_{ni}^2}/\Delta f} = 1.0\,\mu\mathrm{V}$ at $f = 5033\,\mathrm{Hz}$.

13. An amplifier is driven by a source which has a value of $X_s = \mathrm{Im}\left[Z_s\right]$ that minimizes $\overline{v_{ni}^2}$. Show that the noise factor is given by

$$F = 1 + \frac{G_n}{R_s}\left(R_{s(opt)}^2 + 2R_sR_\gamma + R_s^2\right)$$

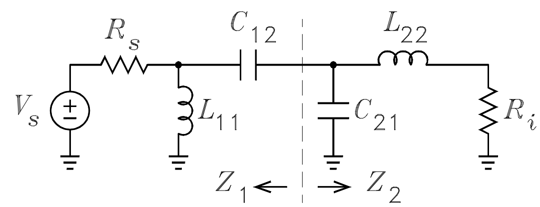
where G_n is the noise conductance defined by $\overline{i_n^2} = 4kT_0G_n\Delta f$, $R_{s(opt)} = \mathrm{Re}\left[Z_{s(opt)}\right]$, and $R_\gamma = \mathrm{Re}\left[Z_\gamma\right] = \mathrm{Re}\left[\gamma\sqrt{\overline{v_n^2}}/\sqrt{\overline{i_n^2}}\right]$.

14. An amplifier has an input resistance $R_i = 25\,\Omega$. It is driven from a source with an output resistance $R_s = 50\,\Omega$.

(a) Design a single-stage lossless matching network to match the source to the amplifier at a frequency 10 MHz. Choose an inductor for X_1 and a capacitor for X_2. Calculate the values of the inductor and capacitor.

(b) A lossless matching network such as the one shown in the figure below is called a two-stage binomial matching network when Z_1 and Z_2 are given by

$$Z_1 = Z_2 = \sqrt{R_s R_i}$$



Design the matching network for the operating frequency of 10 MHz. Calculate the values of the inductors and capacitors.

(c) Use SPICE or a math program to obtain a plot of the magnitude of the reflection coefficient seen by the source as a function of frequency for $V_s = 1\,\mathrm{V}$ rms for parts (a) and (b). The reflection coefficient is given by

$$\Gamma = \frac{Z_{in} - R_s}{Z_{in} + R_s}$$

where Z_{in} is the impedance seen looking to the right of R_s. From the plots, determine the bandwidth of each network for which $|\Gamma| \leq 0.15$. Compare the bandwidths of the two networks.

(d) The noise parameters of the amplifier are $\sqrt{v_n^2/\Delta f} = 6.32 \times 10^{-10}\,\mathrm{V}/\sqrt{\mathrm{Hz}}$, $\sqrt{i_n^2/\Delta f} = 1.79 \times 10^{-11}\,\mathrm{V}/\sqrt{\mathrm{Hz}}$, and $\rho = 0.354$. Calculate $\sqrt{v_{ni}^2/\Delta f}$, F, and NF.