# CHAPTER III

# BOOLEAN ALGEBRA

# BOOLEAN VALUES

## INTRODUCTION



- Boolean algebra is a form of algebra that deals with single digit binary values and variables.
- Values and variables can indicate some of the following binary pairs of values:
  - ON / OFF
  - TRUE / FALSE
  - HIGH / LOW
  - CLOSED / OPEN
  - 1 / 0

# BOOL. OPERATIONS

## FUNDAMENTAL OPERATORS



- Three fundamental operators in Boolean algebra
  - **NOT**: unary operator that complements represented as $\overline{\mathbf{A}}$, $\mathbf{A}'$, or $\sim\mathbf{A}$
  - **AND**: binary operator which performs logical multiplication
    - i.e. **A** ANDed with **B** would be represented as $\mathbf{AB}$ or $\mathbf{A} \cdot \mathbf{B}$
  - **OR**: binary operator which performs logical addition
    - i.e. **A** ORed with **B** would be represented as $\mathbf{A} + \mathbf{B}$

**NOT**

| A | $\overline{\mathbf{A}}$ |
|---|---|
| 0 | 1 |
| 1 | 0 |

**AND**

| A | B | AB |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

**OR**

| A | B | A + B |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

# BOOL. OPERATIONS

## BINARY BOOLEAN OPERATORS



- Below is a table showing all possible Boolean functions $\mathbf{F_N}$ given the two-inputs **A** and **B**.

| A | B | $F_0$ | $F_1$ | $F_2$ | $F_3$ | $F_4$ | $F_5$ | $F_6$ | $F_7$ | $F_8$ | $F_9$ | $F_{10}$ | $F_{11}$ | $F_{12}$ | $F_{13}$ | $F_{14}$ | $F_{15}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |

- $F_0$: $0$ — **Null**
- $F_1$: $AB$
- $F_2$: **Inhibition**
- $F_3$: $A$
- $F_4$: **Inhibition**
- $F_5$: $B$
- $F_6$: $A \oplus B$
- $F_7$: $A + B$
- $F_8$: $\overline{A + B}$
- $F_9$: $\overline{A \oplus B}$
- $F_{10}$: $\overline{B}$
- $F_{11}$: **Implication**
- $F_{12}$: $\overline{A}$
- $F_{13}$: **Implication**
- $F_{14}$: $\overline{AB}$
- $F_{15}$: $1$ — **Identity**

# BOOLEAN ALGEBRA

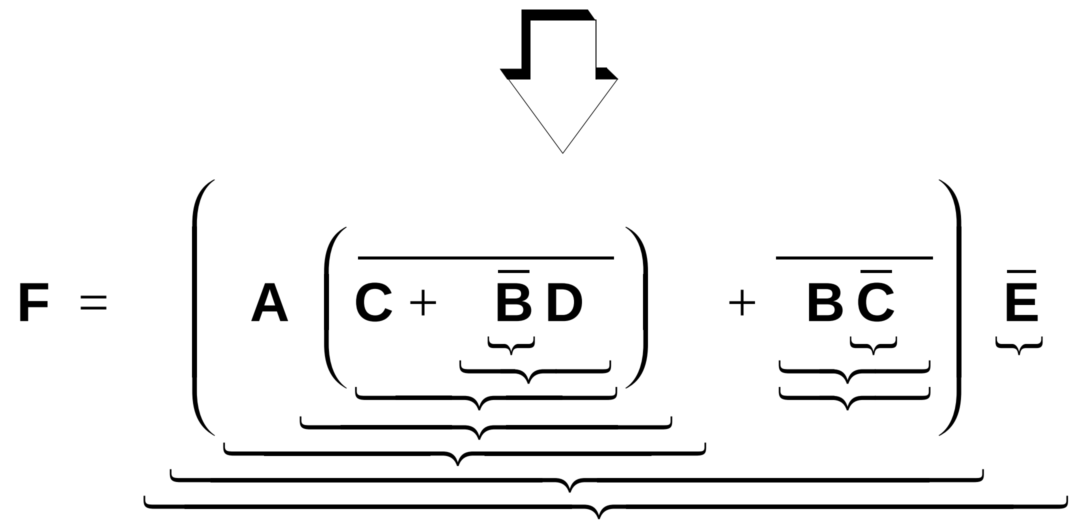
## PRECEDENCE OF OPERATORS



- Boolean expressions must be evaluated with the following order of operator precedence
  - parentheses
  - NOT
  - AND
  - OR

Example:

$$F = \left(A\left(\overline{C + \overline{B}D}\right) + \overline{B\overline{C}}\right)\overline{E}$$



# BOOLEAN ALGEBRA

## FUNCTION EVALUATION



- Example 1:

  Evaluate the following expression when $\mathbf{A} = \mathbf{1}$, $\mathbf{B} = \mathbf{0}$, $\mathbf{C} = \mathbf{1}$

  $$\mathbf{F} = \mathbf{C} + \overline{\mathbf{C}}\mathbf{B} + \mathbf{B}\overline{\mathbf{A}}$$

  - Solution

  $$\mathbf{F} = \mathbf{1} + \overline{\mathbf{1}} \cdot \mathbf{0} + \mathbf{0} \cdot \overline{\mathbf{1}} = \mathbf{1} + \mathbf{0} + \mathbf{0} = \mathbf{1}$$

- Example 2:

  Evaluate the following expression when $\mathbf{A} = \mathbf{0}$, $\mathbf{B} = \mathbf{0}$, $\mathbf{C} = \mathbf{1}$, $\mathbf{D} = \mathbf{1}$

  $$\mathbf{F} = \mathbf{D}(\mathbf{B}\overline{\mathbf{C}}\mathbf{A} + \overline{(\mathbf{A}\overline{\mathbf{B}} + \mathbf{C})} + \mathbf{C})$$

- Solution

  $$\mathbf{F} = \mathbf{1} \cdot (\mathbf{0} \cdot \overline{\mathbf{1}} \cdot \mathbf{0} + \overline{(\mathbf{0} \cdot \overline{\mathbf{0}} + \mathbf{1})} + \mathbf{1}) = \mathbf{1} \cdot (\mathbf{0} + \overline{\mathbf{1}} + \mathbf{1}) = \mathbf{1} \cdot \mathbf{1} = \mathbf{1}$$

# BOOLEAN ALGEBRA

## BASIC IDENTITIES



| | | |
|---|---|---|
| $X + 0 = X$ | $X \cdot 1 = X$ | Identity |
| $X + 1 = 1$ | $X \cdot 0 = 0$ | |
| $X + X = X$ | $X \cdot X = X$ | Idempotent Law |
| $X + X' = 1$ | $X \cdot X' = 0$ | Complement |
| $(X')' = X$ | | Involution Law |
| $X + Y = Y + X$ | $XY = YX$ | Commutativity |
| $X + (Y + Z) = (X + Y) + Z$ | $X(YZ) = (XY)Z$ | Associativity |
| $X(Y + Z) = XY + XZ$ | $X + YZ = (X + Y)(X + Z)$ | Distributivity |
| $X + XY = X$ | $X(X + Y) = X$ | Absorption Law |
| $X + X'Y = X + Y$ | $X(X' + Y) = XY$ | Simplification |
| $(X + Y)' = X'Y'$ | $(XY)' = X' + Y'$ | DeMorgan's Law |
| $XY + X'Z + YZ = XY + X'Z$ | $(X + Y)(X' + Z)(Y + Z) = (X + Y)(X' + Z)$ | Consensus Theorem |

# BOOLEAN ALGEBRA

## DUALITY PRINCIPLE



- **Duality principle:**
  - States that a Boolean equation remains valid if we take the dual of the expressions on both sides of the equals sign.
  - The dual can be found by interchanging the **AND** and **OR** operators along with also interchanging the **0**'s and **1**'s.
  - This is evident with the duals in the basic identities.
    - For instance: DeMorgan's Law can be expressed in two forms

$(\mathbf{X}+\mathbf{Y})' = \mathbf{X}'\mathbf{Y}'$ as well as $(\mathbf{X}\mathbf{Y})' = \mathbf{X}'+\mathbf{Y}'$

# BOOLEAN ALGEBRA

## FUNCTION MANIPULATION (1)



- Example: Simplify the following expression

$$\mathbf{F} = \mathbf{BC} + \mathbf{B\overline{C}} + \mathbf{BA}$$

- Simplification

$$\mathbf{F} = \mathbf{B}(\mathbf{C} + \overline{\mathbf{C}}) + \mathbf{BA}$$

$$\mathbf{F} = \mathbf{B} \cdot \mathbf{1} + \mathbf{BA}$$

$$\mathbf{F} = \mathbf{B}(\mathbf{1} + \mathbf{A})$$

$$\mathbf{F} = \mathbf{B}$$

# BOOLEAN ALGEBRA

## FUNCTION MANIPULATION (2)



- Example: Simplify the following expression

$$F = A + \overline{A}B + \overline{A}\,\overline{B}C + \overline{A}\,\overline{B}\,\overline{C}D + \overline{A}\,\overline{B}\,\overline{C}\,\overline{D}E$$

- Simplification

$$F = A + \overline{A}(B + \overline{B}C + \overline{B}\,\overline{C}D + \overline{B}\,\overline{C}\,\overline{D}E)$$

$$F = A + B + \overline{B}C + \overline{B}\,\overline{C}D + \overline{B}\,\overline{C}\,\overline{D}E$$

$$F = A + B + \overline{B}(C + \overline{C}D + \overline{C}\,\overline{D}E)$$

$$F = A + B + C + \overline{C}D + \overline{C}\,\overline{D}E$$

$$F = A + B + C + \overline{C}(D + \overline{D}E)$$

$$F = A + B + C + D + \overline{D}E$$

$$F = A + B + C + D + E$$

# BOOLEAN ALGEBRA

## FUNCTION MANIPULATION (3)



- Example: Show that the following equality holds

$$\overline{A(\bar{B}\bar{C} + BC)} = \bar{A} + (B + C)(\bar{B} + \bar{C})$$

- Simplification

$$\overline{A(\bar{B}\bar{C} + BC)} = \bar{A} + \overline{(\bar{B}\bar{C} + BC)}$$

$$= \bar{A} + (\overline{\bar{B}\bar{C}})(\overline{BC})$$

$$= \bar{A} + (B + C)(\bar{B} + \bar{C})$$

# STANDARD FORMS

## SOP AND POS



- Boolean expressions can be manipulated into many forms.
- Some standardized forms are required for Boolean expressions to simplify communication of the expressions.
  - **Sum-of-products** (**SOP**)
    - Example:

$$\mathbf{F}(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) = \mathbf{A}\mathbf{B} + \overline{\mathbf{B}}\mathbf{C}\overline{\mathbf{D}} + \mathbf{A}\mathbf{D}$$

  - **Products-of-sums** (**POS**)
    - Example:

$$\mathbf{F}(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) = (\mathbf{A}+\mathbf{B})(\overline{\mathbf{B}}+\mathbf{C}+\overline{\mathbf{D}})(\mathbf{A}+\mathbf{D})$$

# STANDARD FORMS

## MINTERMS



- The following table gives the minterms for a **three-input** system

| | | | $m_0$ | $m_1$ | $m_2$ | $m_3$ | $m_4$ | $m_5$ | $m_6$ | $m_7$ |
|---|---|---|---|---|---|---|---|---|---|---|
| **A** | **B** | **C** | $\bar{A}\bar{B}\bar{C}$ | $\bar{A}\bar{B}C$ | $\bar{A}B\bar{C}$ | $\bar{A}BC$ | $A\bar{B}\bar{C}$ | $A\bar{B}C$ | $AB\bar{C}$ | $ABC$ |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

# STANDARD FORMS

## SUM OF MINTERMS



- **Sum-of-minterms** standard form expresses the Boolean or switching expression in the form of a **sum of products** using **minterms**.
  - For instance, the following Boolean expression using minterms

$$\mathbf{F}(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \overline{\mathbf{A}}\,\overline{\mathbf{B}}\,\overline{\mathbf{C}} + \overline{\mathbf{A}}\,\overline{\mathbf{B}}\mathbf{C} + \mathbf{A}\overline{\mathbf{B}}\,\overline{\mathbf{C}} + \mathbf{A}\overline{\mathbf{B}}\mathbf{C}$$

> Minterms are products, so this is called a "sum of products" (SOP)

could instead be expressed as

$$\mathbf{F}(\mathbf{A}, \mathbf{B}, \mathbf{C}) = m_0 + m_1 + m_4 + m_5$$

or more compactly

$$\mathbf{F}(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \sum m(0, 1, 4, 5) = \textit{one-set}(0, 1, 4, 5)$$

# STANDARD FORMS

## MAXTERMS



- The following table gives the maxterms for a **three-input** system

| A | B | C | $M_0$ $A + B + C$ | $M_1$ $A + B + \overline{C}$ | $M_2$ $A + \overline{B} + C$ | $M_3$ $A + \overline{B} + \overline{C}$ | $M_4$ $\overline{A} + B + C$ | $M_5$ $\overline{A} + B + \overline{C}$ | $M_6$ $\overline{A} + \overline{B} + C$ | $M_7$ $\overline{A} + \overline{B} + \overline{C}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |

# STANDARD FORMS

## PRODUCT OF MAXTERMS



- **Product-of-maxterms** standard form expresses the Boolean or switching expression in the form of **product of sums** using **maxterms**.
  - For instance, the following Boolean expression using maxterms

$$\mathbf{F}(\mathbf{A}, \mathbf{B}, \mathbf{C}) = (\mathbf{A} + \mathbf{B} + \overline{\mathbf{C}})(\overline{\mathbf{A}} + \mathbf{B} + \mathbf{C})(\overline{\mathbf{A}} + \overline{\mathbf{B}} + \overline{\mathbf{C}})$$

could instead be expressed as

$$\mathbf{F}(\mathbf{A}, \mathbf{B}, \mathbf{C}) = M_1 \cdot M_4 \cdot M_7$$

or more compactly as

$$\mathbf{F}(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \prod M(1, 4, 7) = \textit{zero-set}(1, 4, 7)$$

# STANDARD FORMS

## MINTERM AND MAXTERM EXP.



- Given an arbitrary Boolean function, such as

$$\mathbf{F}(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \mathbf{AB} + \overline{\mathbf{B}}(\overline{\mathbf{A}} + \overline{\mathbf{C}})$$

how do we form the canonical form for:

- **sum-of-minterms**
  - Expand the Boolean function into a sum of products. Then take each term with a missing variable $\mathbf{X}$ and **AND** it with $\mathbf{X} + \overline{\mathbf{X}}$.
- **product-of-maxterms**
  - Expand the Boolean function into a product of sums. Then take each factor with a missing variable $\mathbf{X}$ and **OR** it with $\mathbf{X}\overline{\mathbf{X}}$.

# STANDARD FORMS

## FORMING SUM OF MINTERMS



- Example

$$\mathbf{F}(\mathbf{A},\mathbf{B},\mathbf{C}) = \mathbf{AB} + \overline{\mathbf{B}}(\overline{\mathbf{A}} + \overline{\mathbf{C}}) = \mathbf{AB} + \overline{\mathbf{A}}\,\overline{\mathbf{B}} + \overline{\mathbf{B}}\,\overline{\mathbf{C}}$$

$$= \mathbf{AB}(\mathbf{C} + \overline{\mathbf{C}}) + \overline{\mathbf{A}}\,\overline{\mathbf{B}}(\mathbf{C} + \overline{\mathbf{C}}) + (\mathbf{A} + \overline{\mathbf{A}})\overline{\mathbf{B}}\,\overline{\mathbf{C}}$$

$$= \overline{\mathbf{A}}\,\overline{\mathbf{B}}\,\overline{\mathbf{C}} + \overline{\mathbf{A}}\,\overline{\mathbf{B}}\mathbf{C} + \mathbf{A}\overline{\mathbf{B}}\,\overline{\mathbf{C}} + \mathbf{AB}\overline{\mathbf{C}} + \mathbf{ABC}$$

$$= \sum m(0, 1, 4, 6, 7)$$

| A | B | C | F | |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 | ← 0 |
| 0 | 0 | 1 | 1 | ← 1 |
| 0 | 1 | 0 | 0 | |
| 0 | 1 | 1 | 0 | |
| 1 | 0 | 0 | 1 | ← 4 |
| 1 | 0 | 1 | 0 | |
| 1 | 1 | 0 | 1 | ← 6 |
| 1 | 1 | 1 | 1 | ← 7 |

Minterms listed as 1s in Truth Table

# STANDARD FORMS

## FORMING PROD OF MAXTERMS



- Example

$$\mathbf{F}(\mathbf{A},\mathbf{B},\mathbf{C}) = \mathbf{AB} + \overline{\mathbf{B}}(\overline{\mathbf{A}} + \overline{\mathbf{C}}) = \mathbf{AB} + \overline{\mathbf{A}}\overline{\mathbf{B}} + \overline{\mathbf{B}}\overline{\mathbf{C}}$$

$$= (\mathbf{A} + \overline{\mathbf{B}})(\mathbf{A} + \overline{\mathbf{B}} + \overline{\mathbf{C}})(\overline{\mathbf{A}} + \mathbf{B} + \overline{\mathbf{C}})$$ (using distributivity)

$$= (\mathbf{A} + \overline{\mathbf{B}} + \mathbf{C}\overline{\mathbf{C}})(\mathbf{A} + \overline{\mathbf{B}} + \overline{\mathbf{C}})(\overline{\mathbf{A}} + \mathbf{B} + \overline{\mathbf{C}})$$

$$= (\mathbf{A} + \overline{\mathbf{B}} + \mathbf{C})(\mathbf{A} + \overline{\mathbf{B}} + \overline{\mathbf{C}})(\overline{\mathbf{A}} + \mathbf{B} + \overline{\mathbf{C}})$$

$$= \prod M(2, 3, 5)$$

Maxterms are sums, so this is called a "product of sums" (POS)

| A | B | C | F | |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 | |
| 0 | 0 | 1 | 1 | |
| 0 | 1 | 0 | 0 | ← 2 |
| 0 | 1 | 1 | 0 | ← 3 |
| 1 | 0 | 0 | 1 | |
| 1 | 0 | 1 | 0 | ← 5 |
| 1 | 1 | 0 | 1 | |
| 1 | 1 | 1 | 1 | |

Maxterms listed as 0s in Truth Table

# Full Solution of Maxterm Example

F(A,B,C) = AB + **B'**(**A'**+**C'**)

Distributivity (OR)  X(Y+Z) = XY+XZ

AB + **A′B′** + **B′C′**

Distributivity (AND) Identity, X + YZ = (X+Y)(X+Z)

(AB + **A′**) * (AB + **B′**) + **B′C′**

Simplification Identity, X + **X'**Y = X + Y

[ (B + **A′**) * (A + **B′**) ] + **B′C′**

Distributivity (AND) Identity, X = [ (B + **A′**) * (A + **B′**) ],   Y = **B′**,   Z = **C′**)

**(** [ (B + **A′**) * (A + **B′**) ] + **B′** **)** * **(** [ (B + **A′**) * (A + **B′**) ] + **C′** **)**

Distributivity (AND) Identity on each ANDed term:
X = **B′** and **C′** (respectively) Y = (B + **A′**), Z = (A + **B′**))

**(** (**A′ +** B + **B′**) * (A + **B′** + **B′**) **)** * **(** (**A′ +** B + **C′**) * (A + **B′** + **C′**) **)**

Idempotent and Identity – drop (**A′ +** B + **B′**), is always 1,   X+X'=1,  X+1=1,  X*1=X also X+X=X

(A + **B′**) * (**A′** + B + **C′**) * (A + **B′** + **C′**)

Expand (A + **B′**) = (A + B' +CC')  since CC'=0, X+0=X

(A + **B′**) = (A + **B′**+ C)*(A + **B′**+ **C′**)

F(A,B,C) = (A + **B′**+ C) (A + **B′**+ **C′**) (**A′ +** B + **C′**)   where (A + **B′** + **C′**) was duplicated, X+X=X.

Maxterms (**1** where A,B, or C is **complemented** in a term):  010, 011, 101 = 2, 3, 5

# General Rule for Expanding SOP or POS to Maxterms or Minterms

If a term is missing one variable, it expands to two minterms or maxterms.

If a term is missing two variables, it expands to four minterms or maxterms.

If a term is missing N variables, it expands to 2^N minterms or maxterms.

Examples for **F(A,B,C,D) -** replicate incomplete term with every possible combination of complemented and non-complemented missing variables:

**Minterms**

ABC = ABCD + ABC**D**'

AB = ABCD + ABC**D'** + AB**C'**D + AB**C'D'**

A = ABCD + ABC**D'** + AB**C'**D + AB**C'D' +** A**B**'CD + A **B**'C**D'** + A **B**'**C'**D + A **B**'**C'D'**

**Maxterms**

A+B+C = (A+B+C+D) (A+B+C+**D')**

A+B = (A+B+C+D) (A+B+C+**D') (**A+B+**C'**+D) (A+B+**C'**+**D')**

A = (A+B+C+D) (A+B+C+**D') (**A+B+**C'**+D) (A+B+**C'**+**D')***
(A + **B**'+C+D) (A+ **B**'+C+**D') (**A+ **B**'+**C'**+D) (A+ **B**'+**C'**+**D')**

# STANDARD FORMS

## CONVERTING MIN AND MAX



- Converting between sum-of-minterms and product-of-maxterms
  - The two are complementary, as seen by the truth tables.
  - To convert interchange the $\sum$ and $\prod$ , then use missing terms.
    - Example: The example from the previous slides

$$\mathbf{F}(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \sum m(0, 1, 4, 6, 7)$$

F'(A,B,C) = SUM m(2,3,5)

is re-expressed as

$$\mathbf{F}(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \prod M(2, 3, 5)$$

F'(A,B,C) = PI M(0,1,4,6,7)

where the numbers 2, 3, and 5 were missing from the minterm representation.

# SIMPLIFICATION

## KARNAUGH MAPS



- Often it is desired to simplify a Boolean function. A quick graphical approach is to use Karnaugh maps.

**2-variable Karnaugh map**

| A \ B | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

$$F = AB$$

**3-variable Karnaugh map**

| A \ BC | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 0 |
| 1 | 0 | 1 | 1 | 1 |

$$F = AB + C$$

**4-variable Karnaugh map**

| AB \ CD | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 1 | 0 | 0 |
| 01 | 0 | 1 | 0 | 0 |
| 11 | 1 | 1 | 1 | 1 |
| 10 | 0 | 1 | 0 | 0 |

$$F = AB + \overline{C}D$$

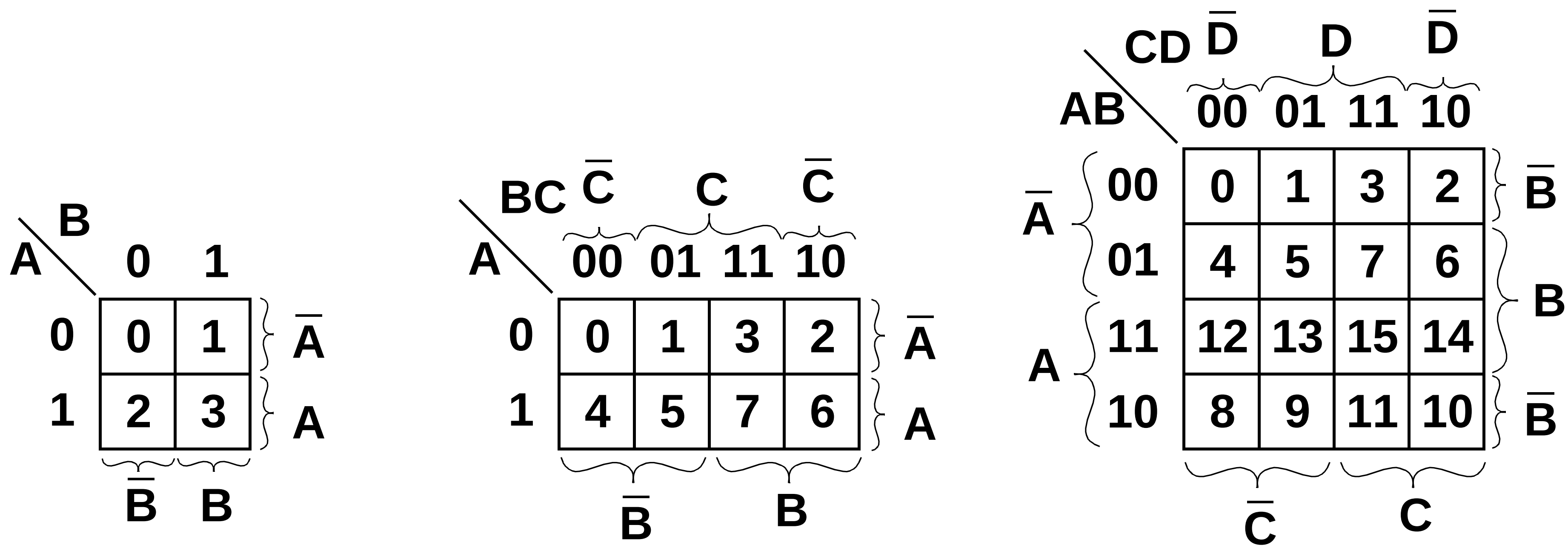
# SIMPLIFICATION

## KARNAUGH MAP ORDERING



- Notice that the ordering of cells in the map are such that moving from one cell to an adjacent cell only changes one variable.

**2-variable Karnaugh map** | **3-variable Karnaugh map** | **4-variable Karnaugh map**



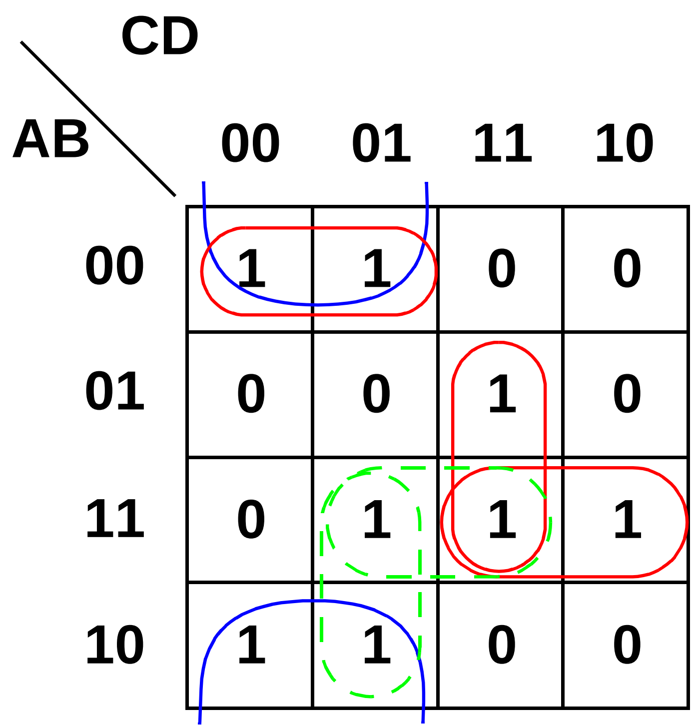

- This ordering allows for grouping of minterms/maxterms for simplification.

# SIMPLIFICATION

## IMPLICANTS



- **Implicant**
  - Bubble covering only **1**'s (size of bubble must be a power of 2).
- **Prime implicant**
  - Bubble that is expanded as big as possible (but increases in size by powers of 2).
- **Essential prime implicant**
  - Bubble that contains a **1** covered only by itself and no other prime implicant bubble.
- **Non-essential prime implicant**
  - All contained **1's** can be covered by another prime implicant bubble.



# SIMPLIFICATION

## PROCEDURE FOR SOP



- **Procedure for finding the SOP from a Karnaugh map**
  - Step 1: Form the 2-, 3-, or 4-variable Karnaugh map as appropriate for the Boolean function.
  - Step 2: Identify all essential prime implicants for **1'**s in the Karnaugh map.
  - Step 3: Identify non-essential prime implicants to cover remaining **1'**s in the Karnaugh map (not covered in step 2).
  - Step 4: For each essential implicant and selected non-essential prime implicants (that cover all 1's), determine the corresponding product term.
  - Step 5: Form a sum-of-products with all product terms from previous step.

# SIMPLIFICATION

## EXAMPLE FOR SOP (1)

SOP: SUM OF PRODUCTS



- Simplify the following Boolean function

$$\mathbf{F}(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \sum m(0, 1, 4, 5) = \bar{\mathbf{A}}\bar{\mathbf{B}}\bar{\mathbf{C}} + \bar{\mathbf{A}}\bar{\mathbf{B}}\mathbf{C} + \mathbf{A}\bar{\mathbf{B}}\bar{\mathbf{C}} + \mathbf{A}\bar{\mathbf{B}}\mathbf{C}$$

- Solution:

| A \ BC | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 |

*zero-set*(*2, 3, 6, 7*)

*one-set*(*0, 1, 4, 5*)

- The essential prime implicants are $\bar{\mathbf{B}}$.
- There are no non-essential prime implicants.
- The sum-of-products solution is $\mathbf{F} = \bar{\mathbf{B}}$.

# SIMPLIFICATION

## EXAMPLE FOR SOP (2)

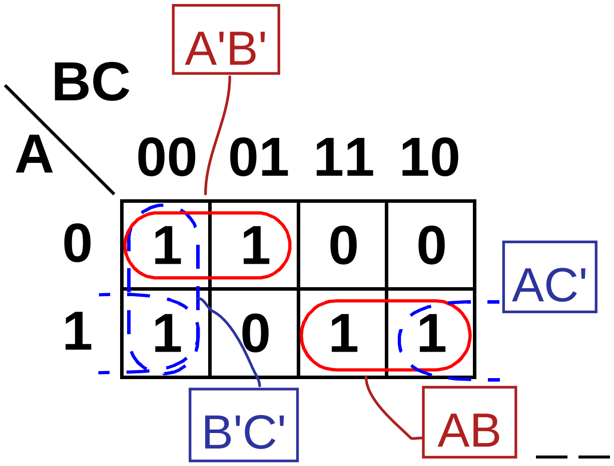
SOP: SUM OF PRODUCTS



- Simplify the following Boolean function

$$\mathbf{F}(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \sum m(0, 1, 4, 6, 7) = \overline{\mathbf{A}}\,\overline{\mathbf{B}}\,\overline{\mathbf{C}} + \overline{\mathbf{A}}\,\overline{\mathbf{B}}\mathbf{C} + \mathbf{A}\overline{\mathbf{B}}\,\overline{\mathbf{C}} + \mathbf{A}\mathbf{B}\overline{\mathbf{C}} + \mathbf{A}\mathbf{B}\mathbf{C}$$

- Solution:




*zero-set*$(2, 3, 5)$

*one-set*$(0, 1, 4, 6, 7)$

- The essential prime implicants are $\overline{\mathbf{A}}\,\overline{\mathbf{B}}$ and $\mathbf{AB}$.
- The non-essential prime implicants are $\overline{\mathbf{B}}\,\overline{\mathbf{C}}$ or $\mathbf{A}\overline{\mathbf{C}}$.
- The sum-of-products solution is

$$\mathbf{F} = \mathbf{AB} + \overline{\mathbf{A}}\,\overline{\mathbf{B}} + \overline{\mathbf{B}}\,\overline{\mathbf{C}} \text{ or } \mathbf{F} = \mathbf{AB} + \overline{\mathbf{A}}\,\overline{\mathbf{B}} + \mathbf{A}\overline{\mathbf{C}}.$$

# SIMPLIFICATION

## PROCEDURE FOR POS

POS: PRODUCT OF SUMS



- **Procedure for finding the SOP from a Karnaugh map**
  - Step 1: Form the 2-, 3-, or 4-variable Karnaugh map as appropriate for the Boolean function.
  - Step 2: Identify all essential prime implicants for **0'**s in the Karnaugh map
  - Step 3: Identify non-essential prime implicants to cover remaining **0'**s in the Karnaugh map.
  - Step 4: For each essential and the selected non-essential prime implicants from each set, determine the corresponding sum term.
  - Step 5: Form a product-of-sums with all sum terms from previous step.

# SIMPLIFICATION

## EXAMPLE FOR POS (1)

POS: PRODUCT OF SUMS



- Simplify the following Boolean function

$$\mathbf{F}(\mathbf{A},\mathbf{B},\mathbf{C}) = \prod M(2,3,5) = (\mathbf{A}+\overline{\mathbf{B}}+\mathbf{C})(\mathbf{A}+\overline{\mathbf{B}}+\overline{\mathbf{C}})(\overline{\mathbf{A}}+\mathbf{B}+\overline{\mathbf{C}})$$

- Solution:

| A \ BC | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 1 | 1 | 0 | 0 |
| 1 | 1 | 0 | 1 | 1 |

*zero-set*$(2, 3, 5)$

*one-set*$(0, 1, 4, 6, 7)$

- The essential prime implicants are $\overline{\mathbf{A}}+\mathbf{B}+\overline{\mathbf{C}}$ and $\mathbf{A}+\overline{\mathbf{B}}$.
- There are no non-essential prime implicants.
- The product-of-sums solution is $\mathbf{F} = (\mathbf{A}+\overline{\mathbf{B}})(\overline{\mathbf{A}}+\mathbf{B}+\overline{\mathbf{C}})$.

# SIMPLIFICATION

## EXAMPLE FOR POS (2)



- Simplify the following Boolean function

$$\mathbf{F}(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \prod M(0, 1, 5, 7, 8, 9, 15)$$

- Solution:
  - The essential prime implicants are $\mathbf{B} + \mathbf{C}$ and $\overline{\mathbf{B}} + \overline{\mathbf{C}} + \overline{\mathbf{D}}$.
  - The non-essential prime implicants can be $\mathbf{A} + \overline{\mathbf{B}} + \overline{\mathbf{D}}$ or $\mathbf{A} + \mathbf{C} + \overline{\mathbf{D}}$.
  - The product-of-sums solution can be either

$$\mathbf{F} = (\mathbf{B} + \mathbf{C})(\overline{\mathbf{B}} + \overline{\mathbf{C}} + \overline{\mathbf{D}})(\mathbf{A} + \overline{\mathbf{B}} + \overline{\mathbf{D}})$$

or

$$\mathbf{F} = (\mathbf{B} + \mathbf{C})(\overline{\mathbf{B}} + \overline{\mathbf{C}} + \overline{\mathbf{D}})(\mathbf{A} + \mathbf{C} + \overline{\mathbf{D}})$$

*zero-set*( *0, 1, 5, 7, 8, 9, 15*)

*one-set*( *2, 3, 4, 6, 10, 11, 12, 13, 14*)

| AB \ CD | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 0 | 1 | 1 |
| 01 | 1 | 0 | 0 | 1 |
| 11 | 1 | 1 | 0 | 1 |
| 10 | 0 | 0 | 1 | 1 |

# SIMPLIFICATION

## DON'T-CARE CONDITION



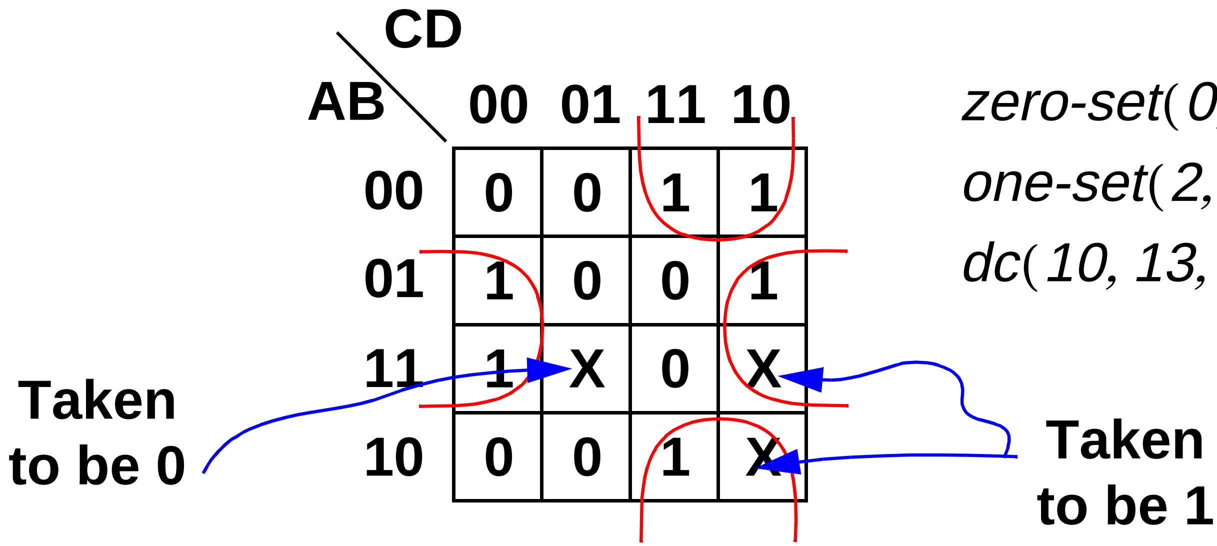
- Switching expressions are sometimes given as **incomplete**, or with **don't-care conditions**.
  - Having don't-care conditions can simplify Boolean expressions and hence simplify the circuit implementation.
  - Along with the *zero-set*( ) and *one-set*( ), we will also have *dc*( ).
  - Don't-cares conditions in Karnaugh maps
    - Don't-cares will be expressed as an "**X**" or "**-**" in Karnaugh maps.
    - Don't-cares can be bubbled along with the **1**s or **0**s depending on what is more convenient and help simplify the resulting expressions.

# SIMPLIFICATION

## DON'T-CARE EXAMPLE (1)



- Find the SOP simplification for the following Karnaugh map




*zero-set*( *0, 1, 5, 7, 8, 9, 15* )

*one-set*( *2, 3, 4, 6, 11, 12* )

*dc*( *10, 13, 14* )

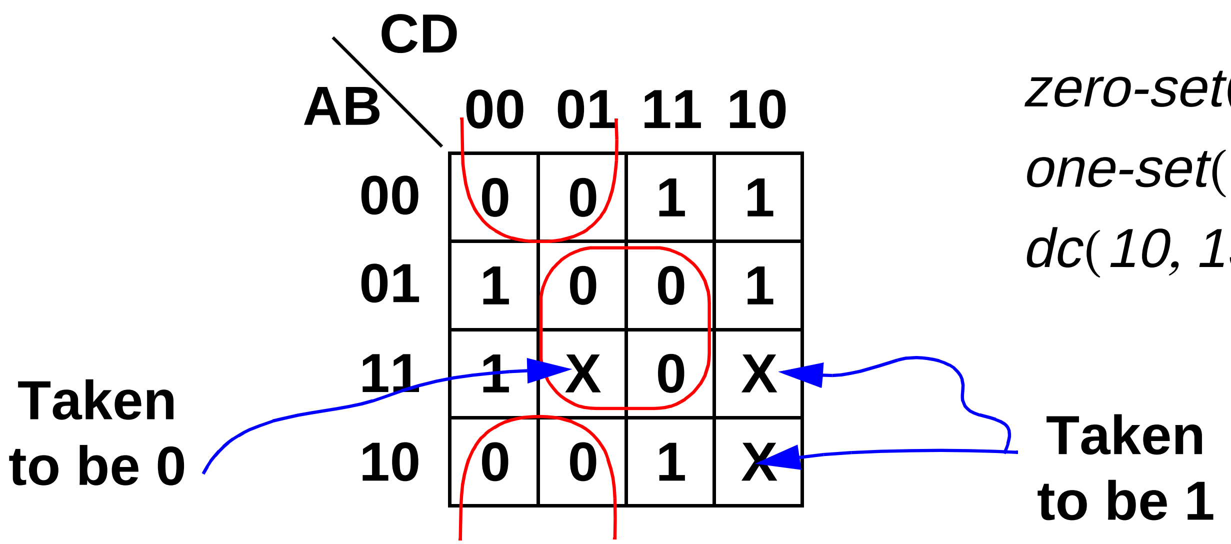
- Solution:
  - The essential prime implicants are $\mathbf{B\bar{D}}$ and $\mathbf{\bar{B}C}$.
  - There are no non-essential prime implicants.
  - The sum-of-products solution is $\mathbf{F} = \mathbf{\bar{B}C} + \mathbf{B\bar{D}}$.

# SIMPLIFICATION

## DON'T-CARE EXAMPLE (2)



- Find the POS simplification for the following Karnaugh map




*zero-set*( *0, 1, 5, 7, 8, 9, 15* )

*one-set*( *2, 3, 4, 6, 11, 12* )

*dc*( *10, 13, 14* )

- Solution:
  - The essential prime implicants are $\mathbf{B} + \mathbf{C}$ and $\bar{\mathbf{B}} + \bar{\mathbf{D}}$.
  - There are no non-essential prime implicants.
  - The product-of-sums solution is $\mathbf{F} = (\mathbf{B} + \mathbf{C})(\bar{\mathbf{B}} + \bar{\mathbf{D}})$.